Examiner and Judge Designs with Multiple Treatments*

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Abstract

Research designs exploiting quasi-random assignment of cases to judges and other examiners are increasingly popular, often using examiner leniency as instruments. Interest is also growing in identifying causal effects with multiple unordered treatments. Identification can benefit from instruments that shift cases toward one treatment without moving cases into any other treatment. Examiner designs, however, do not directly provide such instruments. We construct them by developing a discrete choice model in which examiners choose the treatment they assess most favorably. Slutsky symmetry of treatment propensities helps identify the model's coefficients without exclusion restrictions on case characteristics, and we propose a semiparametrically efficient estimator of these coefficients. Predicted assessments provide instruments for nonparametrically identifying mean potential outcomes and pairwise treatment effects at margins between two treatments. We develop sieve estimation and inference procedures for the treatment effects and establish their asymptotic properties. Revisiting a recent study of Virginia courts, we obtain different findings on how noncarceral and carceral convictions affect reoffending.

Keywords: Leniency design, multivalued treatments, semiparametric efficiency, marginal treatment effects.

JEL codes: C14, C21, C25, C26.

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1 Introduction

Many high-stakes treatments are assigned by expert examiners such as judges, patent examiners and child welfare investigators: a judge dismisses the charges, convicts a defendant without incarceration, or incarcerates the defendant; a patent examiner grants an application as filed, grants it with narrowed claims, or rejects it; a child welfare investigator closes a case, refers the family to community services, or removes the child. Examiners affect later outcomes through the treatments they assign, and the effects of these treatments are of interest. Feng and Jaravel (2020) ask how narrowing the claims affects later litigation, and Gross and Baron (2022) how removing the child affects safety and schooling. Such questions are difficult to answer because cases sort into different treatments based on unobserved heterogeneity correlated with outcomes. Examiner designs address this difficulty by exploiting the quasi-random assignment of cases to examiners, and measures of examiner leniency are often used as instruments. These designs typically rely on the binary treatment framework by collapsing the treatment categories into the focal treatment and a composite of the remaining treatments. Chyn, Frandsen, and Leslie (2025) survey examiner and judge designs, and Goldsmith-Pinkham, Hull, and Kolesár (2026) give a practical guide to estimation and inference with them.

Collapsing treatments in this way can obscure the differences between them, because different treatments influence outcomes through distinct mechanisms. Our leading example comes from Humphries et al. (2025), who ask how a noncarceral conviction, relative to dismissal or to incarceration, affects defendants' reoffending. Noncarceral conviction may affect reoffending through a criminal record, whereas incarceration may affect it through incapacitation. In such settings, it is useful to identify the causal effects of one treatment relative to another for compliers moved between the two treatments. The natural candidates for instruments are leniency instruments, given by each examiner's propensity for each treatment. To identify the pairwise effect between two treatments, researchers would ideally use changes in instruments that each draw cases into one treatment of the pair only from the other. With multiple unordered treatments, flows between treatments can run in any direction. Researchers then generally cannot tell which changes in the leniency instruments meet this requirement, because the observed propensities for each treatment are net shares and do not pin down these flows.

In the context of educational attainment, Mountjoy (2022, Eqs. (11)–(14)) identifies the pairwise effects among no college, a two-year college, and a four-year college, using the

distances to the nearest two-year and four-year colleges as continuous instruments. Each distance moves students into one type of college without substitution between the other two options. Such instruments are, however, rarely available in examiner settings. This paper constructs these instruments from a model of examiners' decisions between three treatments $t \in \{0, 1, 2\}$:

$$T(j) = \underset{t \in \{0,1,2\}}{\arg\max} \begin{cases} 0 & t = 0 \\ X'\gamma_j^1 - U_j^1 & t = 1 \\ X'\gamma_j^2 - U_j^2 & t = 2 \end{cases}.$$

Examiner $j \in \{1, \ldots, J\}$ assesses each treatment against the baseline treatment 0 and assigns the treatment that maximizes her assessment. The assessment of treatment t is a linear combination of the observed case characteristics X with coefficients γ_j^t (including an intercept), minus an unobserved component U_j^t. We require X to contain at least two continuous characteristics, and the coefficients may vary freely across the examiners. The unobserved components are identically distributed across cases and examiners but otherwise unrestricted. We use the assigned examiner's predicted assessments as instruments. Under quasi-randomization of examiners, these instruments are exogenous conditional on the case characteristics. As with the distances in Mountjoy (2022), increasing one instrument while keeping the other fixed moves cases into its treatment without inducing selection into the others.

Suppose that for each case a researcher observes the outcome Y, the treatment T, the case characteristics X, and the assigned examiner. Our primary contribution is to identify examiner-specific coefficients and unknown propensity functions when the same case characteristics enter both assessments. Write Z^t for the predicted assessment of treatment t and $Z = (Z^1, Z^2)$. The propensity function P^t for treatment t is defined as $P^t(z) = \mathbb{P}(T = t \,|\, Z = z)$ with $z = (z^1, z^2)$, which is determined by the unknown distribution of the unobserved components. Theorem 1 establishes the identification with at least two continuous characteristics.

The key to our identification argument is a symmetry in how the propensities respond to the predicted assessments as aggregates of the characteristics. The propensities satisfy the Slutsky symmetry $\partial P^1/\partial z^2 = \partial P^2/\partial z^1$ because a marginal increase in z^2 draws the same cases from 1 into 2 as a marginal increase in z^1 draws from 2 into 1. This substitution at the margin is analogous to symmetric cross-price effects in demand without income effects. Examiner j's propensity at characteristics x is the propensity function

evaluated at $(z^1, z^2) = (x'\gamma_j^1, x'\gamma_j^2)$. The responses of her propensities to the two continuous characteristics $x = (x_1, x_2)$ satisfy

$$\begin{pmatrix} \frac{\partial P^1}{\partial x_1} & \frac{\partial P^1}{\partial x_2} \\ \frac{\partial P^2}{\partial x_1} & \frac{\partial P^2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial P^1}{\partial z^1} & \frac{\partial P^1}{\partial z^2} \\ \frac{\partial P^2}{\partial z^1} & \frac{\partial P^2}{\partial z^2} \end{pmatrix} \begin{pmatrix} \gamma_{j,1}^1 & \gamma_{j,2}^1 \\ \gamma_{j,1}^2 & \gamma_{j,2}^2 \end{pmatrix} \Rightarrow \begin{pmatrix} \frac{\partial P^1}{\partial x_1} & \frac{\partial P^1}{\partial x_2} \\ \frac{\partial P^2}{\partial x_1} & \frac{\partial P^2}{\partial x_2} \end{pmatrix} \begin{pmatrix} \gamma_{j,1}^1 & \gamma_{j,2}^1 \\ \gamma_{j,1}^2 & \gamma_{j,2}^2 \end{pmatrix}^{-1} \text{ is symmetric,}$$

where $\gamma_{j,l}^t$ is the coefficient on x_l in z^t and the coefficient matrix is invertible. The matrix on the rightmost side must be symmetric at every characteristic, and any perturbation to the coefficient matrix causes this symmetry to fail at some characteristic. This pins down the coefficients of each examiner. The scale of each examiner's coefficients is determined by the common distribution of the unobserved components. Our result extends semiparametric identification of multinomial choice models to settings with only common regressors and a nonparametric distribution of unobserved heterogeneity.

Our identification result does not follow from existing results. The index structure identifies only the set of linear combinations of γ_j^1 and γ_j^2, not each coefficient vector. Prior work identifies the coefficients from an excluded regressor for each treatment (Matzkin, 1993; Lewbel, 2000; Ichimura and Lee, 1991) or fixes them by a normalization (Berry and Haile, 2024). Neither is suited to our setting. An excluded regressor enters the assessment of one treatment alone and typically measures the cost of taking up that treatment. Examiner designs rarely provide such variables. Normalized coefficients are chosen by the researcher and need not coincide with the true coefficients that our instruments require. The closest work is Allen and Rehbeck (2019), who use the Slutsky symmetry to dispense with large support in settings with excluded regressors for each treatment. We use the symmetry to dispense with both excluded regressors and large support.

Our model allows examiners to respond differently to the case characteristics. The ranking of examiners by their predicted assessment of a treatment can then vary with the characteristics. Humphries et al. (2025) construct their instruments from an unordered model that lets examiners differ in the intercepts of their assessments and share the slopes on the case characteristics. Rivera (2023) uses examiner-specific coefficients in an ordered model to enlarge the support of the propensity scores and to require monotonicity only within each covariate cell. In our application, common coefficients are rejected for most case characteristics.

Our second contribution is a nonparametric identification formula for mean potential outcomes at margins between two treatments. To fix ideas, consider the margin between treatments 1 and 2. For compliers moved between treatments 1 and 2, we identify the

mean of the potential outcome under $t \neq 0$, $Y(t)$, given characteristics x as (Proposition 1)

$$\begin{aligned} &\mathbb{E}\left[Y(t) \,|\, X = x, z^1 - U^1 = z^2 - U^2 > 0\right] \\ &= \frac{\partial}{\partial p}\Big(p\mathbb{E}\left[Y \,|\, T = t, X = x, P^t(Z) = p, Z^t = z^t\right]\Big)_{p=P^t(z)}, \end{aligned} \tag{1}$$

where (U^1, U^2) are the assigned examiner's unobserved components. Holding the predicted assessment of treatment t fixed ensures that the cases newly receiving treatment t all come from the other treatment of the pair, not from 0. Each mean requires one regression and its partial derivative in the propensity, whereas existing methods divide a derivative of the outcome-by-treatment regression by a derivative of the propensity (Lee and Salanié, 2018; Mountjoy, 2022). Estimation of such ratios can be unstable when the denominator is small.

From these means, we identify the pairwise treatment effect for the subpopulation at the margin between two treatments. Such margin-specific effects are relevant when a policy change expands or contracts one treatment along its margins with the other two, such as a revision of sentencing guidelines that discourages incarceration. The overall effect of the policy is then a weighted average of the two margin-specific effects, with weights identified by the shares of cases drawn from each margin. Our margin-specific effects are for compliers between two treatments, whereas the marginal treatment effect (MTE) of Lee and Salanié (2018) is for compliers with $(U^1, U^2) = (z^1, z^2)$. Standard policy-relevant parameters can be identified by aggregating the MTE (Heckman and Vytlacil, 2005). Our method identifies the MTE as well, by taking one further derivative of each mean potential outcome.

The formula requires that the instruments have continuous support given the characteristics, whereas they take at most as many values as there are examiners at fixed characteristics. We specify the regression in the formula as linear in the characteristics, with coefficients that vary with the propensity and the index difference. Under this varying-coefficient specification, the pairwise effects are identified with finitely many examiners if the characteristics have full rank at each value of the instruments. Brinch et al. (2017) address the same type of problem for a binary treatment with a discrete instrument by restricting the functional form of the marginal treatment effect. We instead restrict how the case characteristics enter the regression and leave its dependence on the instruments unrestricted.

Estimation proceeds in two stages. The first stage estimates the discrete choice model

from case-level data by sieve maximum likelihood, and the estimator is semiparametrically efficient for the examiner-specific coefficients. The first-stage estimates of the coefficients and the propensity functions construct the instruments for the second stage. The second stage estimates the outcome regressions in (1) by sieve least squares under the varying-coefficient specification. The generated instruments attenuate the regression coefficients, and we correct for this bias. The pairwise treatment effects follow by plugging the corrected estimates into (1) and differencing the means. We provide asymptotic standard errors for their functionals, including aggregated effects.

We revisit Humphries et al. (2025) and illustrate our method by applying it to felony cases in the Virginia Circuit Courts. We measure recidivism by new felony charges, convictions and carceral sentences, in four time windows after disposition. Relative to dismissal, noncarceral conviction has mostly positive but statistically insignificant effects, and the increase in reoffending within seven years after disposition is primarily noncarceral. Incarceration relative to dismissal has the largest effects, significant at the five percent level from the second year on, and the increase consists almost entirely of new carceral sentences. Incarceration relative to noncarceral conviction has effects mostly indistinguishable from zero. Over the first seven years, our estimates of the effects of noncarceral conviction relative to dismissal on each recidivism measure are less than half of the two-stage least squares estimates that Humphries et al. (2025, Table IV) obtain from the leniency instruments. Ours are smaller in the first four years and larger in years five to seven. Their study does not report effects of incarceration relative to dismissal. Our estimates also suggest heterogeneity across case characteristics.

We conduct simulation studies calibrated to the application. We compare our estimator with its counterpart that estimates the examiner-coefficients with logistic regression. The second-stage estimation is the same except that the instruments are constructed from the logistic first stage. Two data-generating processes (DGPs) are considered, one with the unobserved components following the logistic distribution and the other with a normal distribution that makes the logit alternative mildly misspecified. In the first DGP, our estimator loses little efficiency relative to the logit in the first stage. The second-stage estimator performs comparably to both the oracle estimator, which knows the true instruments in the second stage, and the logit alternative. In the second DGP, our estimator remains comparable to the oracle in the second stage, whereas the logit alternative is biased on some margins. Confidence intervals based on our estimator show coverage close to the nominal level in both DGPs.

The rest of the paper is organized as follows. Section 2 presents the first-stage model of examiners' decision-making, defines the margin-specific effects of interest, and establishes their identification. It also specifies the outcome regressions of the second stage. Section 3 establishes identification of the first-stage parameters. Section 4 develops the two-stage procedure, the sieve maximum likelihood estimation of the first stage that recovers the examiners' indices and the varying-coefficient second stage that estimates the margin-specific effects. Section 5 establishes the large-sample properties of both stages. Section 6 applies the framework to felony cases in the Virginia Circuit Courts. Section 7 evaluates both stages in Monte Carlo experiments calibrated to the application. Section 8 concludes. All technical proofs are collected in the Appendix.

2 Econometric Framework

We focus on settings with three mutually exclusive and exhaustive treatments $t \in \{0, 1, 2\}$. Our method is applied to Virginia courts in Section 6, and in this setting $t = 0$ is a dismissal, $t = 1$ is a noncarceral conviction, and $t = 2$ is incarceration. The identification results in this paper should extend to settings with more than three treatments under suitable modifications, but we restrict attention to three treatments for clarity of exposition.

Let i index cases and j index examiners. Each case is randomly assigned to a single examiner, who assigns the treatment for that case. Let $T_i \in \{0, 1, 2\}$ denote the observed treatment for case i, X_i a vector of observed case characteristics of dimension d_X whose last component is the constant one, J_i the examiner assigned to case i, and Y_i an outcome of interest. The data $(Y_i, T_i, X_i, J_i)_{i=1}^N$ are available for N cases handled by J examiners. For $t \in \{0, 1, 2\}$, we denote the corresponding treatment indicator by $T_i^t = \mathbb{1}\{T_i = t\}$. Let $Y_i(t, j)$ denote the potential outcome of case i under treatment t and examiner j.

2.1 Unordered Treatment Assignment Model

Let $T_i(j)$ be the treatment that case i would receive when assigned to examiner j, so that the realized treatment is observed as $T_i = T_i(J_i)$. We posit that the examiner assesses each treatment on observed case-level characteristics and unobserved heterogeneity, and that the assigned treatment is the one with the highest assessment. We assume that the

treatment under examiner j takes the following form:

$$T_i(j) = \underset{t\in\{0,1,2\}}{\arg\max}\,(X_i'\gamma_j^t - U_{ij}^t), \tag{2}$$

where γ_j^t is a vector of examiner-specific parameters entering the assessment of treatment t and $U_{ij} = (U_{ij}^1, U_{ij}^2)$ is a vector of unobserved treatment-specific heterogeneity of case i under examiner j. The assessment of the base treatment 0 is normalized to zero across all cases and examiners by setting $\gamma_j^0 \equiv 0$ for all j and $U_{ij}^0 \equiv 0$ for all i, without loss of generality. Each assessment is thus measured relative to the base treatment 0, and we maintain this convention throughout the paper unless otherwise noted. We write $U_i = U_{iJ_i}$ for the unobserved components of case i under the assigned examiner, so that the realized treatment T_i maximizes $X_i'\gamma_{J_i}^t - U_i^t$ over t.

The assessment of treatment $t \in \{1, 2\}$ is the predictable component $X_i'\gamma_j^t$ less the unobserved component U_{ij}^t. We refer to the predictable component as the examiner index for treatment t, which captures the examiner's mean assessment of treatment t for case i. The assumption is that each examiner index is linear in observed case characteristics with examiner-specific intercepts and slope coefficients. The intercept γ_{j,d_X}^t captures examiner j's baseline assessment, and the slopes on the non-intercept characteristics capture the examiner's differential responsiveness to these case characteristics.

Let $\mathcal{F}$ be the collection of continuous densities on $\mathbb{R}^2$ that are positive at every point and satisfy $f(u) \le C(1 + \|u\|)^{-(1+\delta)}$ for all $u \in \mathbb{R}^2$ and some $C < \infty$ and $\delta > 0$.

Assumption 1. (Treatment Assignment Model). *The following hold.*

(a) (Random assignment.) J_i *is independent of* $\{X_i, (U_{ij})_{j=1}^J, (Y_i(t,j))_{t\in\{0,1,2\},j\in\{1,\ldots,J\}}\}$.

(b) (Assessment model.) *For every examiner* j, $T_i(j)$ *is given by* (2), *and* $T_i = T_i(J_i)$.

(c) (Unobserved heterogeneity.) *For every examiner* j, U_{ij} *is independent of* X_i *and has a density* $f \in \mathcal{F}$ *that does not depend on* j.

Assumption 1(a) treats the assignment of examiners to cases as completely random. In practice, assignment may be random only within cells defined by court or court-by-year, and the researcher can account for this by including the cell indicators among the controls, as we do in Section 6.

Two restrictions on the unobserved heterogeneity are built into Assumption 1. First, the distribution of U_{ij} is the same for every examiner j, while the dependence of $(U_{ij})_{j=1}^J$

across examiners is unrestricted. The substantive requirement is that examiners face a common distribution of the unobserved components, not that the realizations coincide examiner by examiner. The function that maps the examiner indices to the treatment probabilities is the same for every examiner.

Second, U_{ij} is independent of the observed case characteristics X_i, and with random assignment U_i is independent of (X_i, J_i). Full independence between U_{ij} and X_i is another substantive restriction. Interpreting U_{ij} as the residual of the assessments after conditioning on X_i motivates mean independence at most, whereas full independence further requires that no feature of the distribution of U_{ij}, such as its dispersion, varies with case characteristics.[1] We impose it because of what it delivers. Under independence, the treatment probabilities conditional on the case and the examiner depend on (X_i, J_i) only through the examiner indices, as shown in (4) below.

In addition, U_{ij} possesses a density f, which is commonly assumed in discrete-choice settings to ensure that the probability of ties is zero. The class $\mathcal{F}$ restricts the continuity, the positivity and the tails of f.

Let the examiner-index instrument be $Z_i = (Z_i^1, Z_i^2)$, where $Z_i^t = X_i'\gamma_{J_i}^t$ for treatment $t \in \{1, 2\}$. Let the *propensity function* for treatment t be defined as

$$P^t(z) := \int_{\mathbb{R}^2} \mathbb{1}\{z^t - u^t \geq z^s - u^s,\, \forall s \neq t\} f(u)du, \qquad z \in \mathbb{R}^2, \quad t \in \{0, 1, 2\}, \tag{3}$$

with the convention $z^0 = u^0 = 0$, and write $P(z) = (P^1(z), P^2(z))$ for the pair.[2] Under the assessment model, treatment t is chosen under z if and only if U_i belongs to the argmax region $\{u \in \mathbb{R}^2 : z^t - u^t \geq z^s - u^s, \forall s \neq t\}$, over which the integral in (3) runs. We call the conditional choice probability $\mathbb{P}(T_i = t \,|\, X_i = x, J_i = j)$ the treatment-t propensity of examiner j at x. Let $\gamma_j = (\gamma_j^1, \gamma_j^2)$ be a $d_X \times 2$ matrix stacking the vectors of j-specific coefficients across treatments 1 and 2. Together, these vectors characterize how examiner j assesses the treatments and are treated as unknown parameters to be estimated from the data. Since U_i is independent of (X_i, J_i) under Assumption 1, the treatment-t propensity depends on (x, j) only through the examiner indices $x'\gamma_j = (x'\gamma_j^1, x'\gamma_j^2)$:

$$\mathbb{P}(T_i = t \,|\, X_i = x, J_i = j) = P^t(x'\gamma_j), \qquad t \in \{0, 1, 2\}. \tag{4}$$

[1] For identification purposes, the independence can be weakened for discrete characteristics, in which case it needs to hold within each cell defined by the discrete characteristics. We instead pool the cells and let the discrete characteristics enter the assessments through their coefficients.

[2] The propensity function for treatment 0 is left out of the pair, since $P^0(z) = 1 - P^1(z) - P^2(z)$.

The density f speaks to the data only through the propensity functions $P^t(\cdot)$ induced by it, so the propensity functions characterize the data distribution as well as f does. The parameters of the first stage can therefore be regarded as the examiner coefficients and the propensity function. A parametric choice of f instead determines the propensity function outright. If the three assessments carry independent type I extreme value disturbances, for instance, (3) reduces to the multinomial logit form

$$P^t_{\text{logit}}(z) = \frac{\exp(z^t)}{1 + \exp(z^1) + \exp(z^2)}, \qquad t \in \{0, 1, 2\}, \tag{5}$$

again under the convention $z^0 = 0$.[3] The examiner coefficients are then identified, but only if the parametric form is correct.

Conversely, f can be recovered from the propensity function. Treatment 0 is chosen at z when $U_i^1 \geq z^1$ and $U_i^2 \geq z^2$, so $P^0(z)$ is the joint complementary cumulative distribution function of U_i at z. Differentiating in the two indices returns the density itself,

$$\frac{\partial^2 P^0(z)}{\partial z^1 \partial z^2} = f(z^1, z^2), \tag{6}$$

so that knowing $P(\cdot)$ on an open set of index values determines f on that set.

The examiner fixed-effects specification is the special case in which only the intercepts γ^t_{j,d_X} vary across examiners while the slope coefficients are common, which is the form of the assessment model of Humphries et al. (2025), whose thresholds, one per treatment, are the analogue of our intercepts. Under common slopes, the ranking of examiners by their predicted assessment of a treatment is the same for every case, and under interactive effects it may change with the case. The model departs from Humphries et al. (2025) on a second front: they specify a logistic distribution perturbed by a correlated normal component for the unobserved components, whereas we specify none.

The closest antecedent, Rivera (2023), also lets the coefficients vary across examiners, but in an ordered model with one unobserved component. There each index enters the propensities through one univariate function, and results from the single-index literature would identify the coefficients without specifying it. In our unordered model each propensity depends on both indices through the joint distribution of U_i.

Once the examiner-specific coefficients are estimated, we use the assigned examiner's

[3] Here, f is the density of $U_i = (U_i^1, U_i^2)$, which the normalization $U_i^0 \equiv 0$ already expresses relative to the base treatment. The form in (5) obtains when the disturbances $\varepsilon^0, \varepsilon^1, \varepsilon^2$ entering the three assessments are independent and type I extreme value, in which case f is the joint density of $(\varepsilon^0 - \varepsilon^1, \varepsilon^0 - \varepsilon^2)$.

indices as instruments. Because each index enters the assessment of one treatment alone, a change in it moves cases into or out of that treatment without substitution among the others. Given the case characteristics, the indices vary only through the random assignment of the examiner.

2.2 Identifying Mean Potential Outcomes at the Margin

The parameters of interest in this paper are the pairwise treatment effects at the margins between two treatments, the average effect of one treatment relative to the other on the cases at that margin. We begin by formulating them within the potential-outcomes framework. We impose the standard exclusion restriction that examiners affect outcomes only through treatment assignment. Under the exclusion restriction, potential outcomes are indexed by treatment alone. The observed outcome can therefore be written as $Y_i = Y_i(T_i)$. We also require that the joint distribution of an examiner's unobserved components and the potential outcomes be the same for every examiner.

Assumption 2. (Exclusion and Invariance). *The following hold.*

(a) (Exclusion.) *For each $t \in \{0, 1, 2\}$ and examiner j, it holds that $Y_i(t, j) = Y_i(t)$ for all cases i.*

(b) (Invariance.) *The joint distribution of $(U_{ij}, Y_i(0), Y_i(1), Y_i(2))$ conditional on X_i is the same for every examiner j.*

Assumption 2(b) extends the common distribution of U_{ij} in Assumption 1(c) to its joint distribution with the potential outcomes. The condition holds, for instance, when U_{ij} is the sum of a case-level component common to all examiners and an idiosyncratic noise that is identically distributed across examiners and is drawn independently of the other variables of the model.

The examiner-index instruments provide exogenous variation in treatment assignment. Since $Z_i = X_i'\gamma_{J_i}$ is formed by interaction between observed case characteristics and heterogeneity in the randomly assigned examiner, this exogeneity holds conditional on X_i. Let $T_i(z)$ denote the treatment case i would receive from an examiner whose index vector is $z = (z^1, z^2)$, the maximizer of the assessments in (2) with z in place of $X_i'\gamma_j$, so that $T_i = T_i(Z_i)$. Under Assumptions 1 and 2,

$$(T_i(z), Y_i(t))_{z \in \text{supp}(Z_i),\, t \in \{0,1,2\}} \perp\!\!\!\perp Z_i \,|\, X_i. \tag{7}$$

The independence in (7) is joint, not marginal: the entire array of potential treatments and potential outcomes, viewed as a single random element indexed by z and t, is independent of Z_i given X_i. Under the assessment model, $T_i(z)$ is a deterministic function of z and U_i, so (7) follows from the independence of $(U_i, Y_i(0), Y_i(1), Y_i(2))$ from Z_i given X_i, and the identification results below condition on events defined jointly by U_i and the potential outcomes. Assumptions 1(a) and 2 imply that independence. Given X_i and $J_i = j$, random assignment makes $(U_i, Y_i(0), Y_i(1), Y_i(2))$ distributed as $(U_{ij}, Y_i(0), Y_i(1), Y_i(2))$ given X_i, and Assumption 2(b) makes that distribution the same for every examiner. Conditional on X_i, variation in Z_i is thus induced solely by the random assignment of J_i. In practice, controlling for X_i is necessary when using Z_i as instruments, since Z_i varies with X_i as well as with the examiner, and only the variation induced by the random assignment of examiners is exogenous.

There are three margins in our setting: the $(1, 0)$ margin, the $(2, 1)$ margin, and the $(2, 0)$ margin. For any pair $t, s \in \{0, 1, 2\}$ with $t \neq s$, the (t, s) margin consists of cases for which the assessments of treatments t and s are equal and both exceed the assessment of the remaining treatment $r \in \{0, 1, 2\} \setminus \{t, s\}$.

Definition 1 (Margin and Margin-Specific Effects)**.** For $t, s \in \{0, 1, 2\}$ with $t \neq s$, let r be the remaining treatment. For each $z \in \mathbb{R}^2$, the (t, s) margin is the set

$$\mathcal{C}_{ts}(z) = \{u \in \mathbb{R}^2 : z^t - u^t = z^s - u^s > z^r - u^r\}.$$

For each $x \in \text{supp}(X_i)$, the margin-specific mean potential outcome under treatment t is

$$\mu_{ts}(z, x) = \mathbb{E}[Y_i(t) \,|\, X_i = x, U_i \in \mathcal{C}_{ts}(z)],$$

and the margin-specific treatment effect function is $\tau_{ts}(z, x) = \mu_{ts}(z, x) - \mu_{st}(z, x)$.

In words, a case is at the (t, s) margin when t and s are its two best treatments and it assesses them equally; the remaining treatment is strictly dominated and hence irrelevant to the choice. The extension beyond three treatments follows similarly: with any number of alternatives, the (t, s) margin ties the top two assessments at treatments t and s and requires every other assessment to fall strictly below. The strict inequality excludes the event that all three assessments tie, and whether this boundary event is included is immaterial. The (t, s) margin is a one-dimensional subset of the support of U_i, while the triple tie is a single point in it. Relative to a one-dimensional manifold with

positive Hausdorff measure, a point has measure zero, and including the triple tie or not does not affect the conditional mean. Since $\mathcal{C}_{st}(z) = \mathcal{C}_{ts}(z)$, the treatment effect function can be written as one conditional mean $\tau_{ts}(z, x) = \mathbb{E}[Y_i(t) - Y_i(s) \,|\, X_i = x, U_i \in \mathcal{C}_{ts}(z)]$.

Related notions of a margin appear in the literature, and ours provides an intermediate level of resolution compared with them. Heckman et al. (2008) consider the margin between one treatment, being the best treatment, and the next-best alternative outside of it. Their comparison aggregates over whichever alternative happens to be next best, whereas our margin fixes the two treatments in comparison. Their margin is decomposed into several of ours, one for each alternative that is next best. Lee and Salanié (2018) consider margins at fixed values of the unobservables $(U_i^1, U_i^2) = (z^1, z^2)$, a finer notion than ours. Their margin is where all three treatments tie, and our margin aggregates a continuum of theirs. Mountjoy (2022) works with essentially the same margin as ours, although his notion of indifference is defined within the educational choice setting.

Pairwise effects on our margin are directly relevant when a policy change expands or contracts one treatment along its margins with the other two, such as a revision of sentencing guidelines that discourages incarceration. The overall effect of the policy is then given by a weighted average of the two pairwise effects, with weights identified by the shares of cases drawn across each margin. The effect on the margin of Lee and Salanié (2018) is the marginal treatment effect (MTE) with three treatments, and a broad range of policy-relevant parameters can be identified by aggregating it (Heckman and Vytlacil, 2005). Our method can also identify their marginal treatment effects by taking one further derivative of each mean potential outcome (Proposition F.1).

Our identification argument writes the mean potential outcome at each margin as a selected-outcome regression plus a single derivative of that regression in the treatment propensity. Fix a margin between treatments t and s, and let $r \in \{0, 1, 2\} \setminus \{t, s\}$ be the remaining treatment. Let $Z_i^{t,r} = Z_i^t - Z_i^r$ be the relative index of treatment t with respect to treatment r, and let $z^{t,r} = z^t - z^r$ be the corresponding value. The variables $Z_i^{s,r}$ and $z^{s,r}$ are defined analogously. We maintain the convention $Z_i^0 \equiv 0$ and $z^0 \equiv 0$. Let $P_i^t = P^t(Z_i)$ denote the treatment-t propensity evaluated at the instrument of case i.

Define the *margin-specific selected-outcome regression* by

$$m_{ts}(p, a, x) = \mathbb{E}[Y_i \,|\, X_i = x, T_i = t, P_i^t = p, Z_i^{t,r} = a], \quad t, s \in \{0, 1, 2\}, \quad t \neq s.$$

This function gives the mean observed outcome for cases selected into treatment t, con-

ditional on the treatment-t propensity, the relative index of t with respect to r, and observed characteristics. However, this regression need not identify the mean potential outcome under treatment t, because cases observed in treatment t remain selectively sorted even after conditioning on $(P_i^t, Z_i^{t,r}, X_i)$. Consequently, differences such as $m_{ts}(p^t, z^{t,r}, x) - m_{st}(p^s, z^{s,r}, x)$ do not by themselves admit a causal interpretation.

The partial derivative of m_{ts} in the propensity argument reflects the discrepancy between the selected-outcome regression and the mean potential outcome under treatment t. This is expressed in the following identity:

$$\partial_p m_{ts}(p, z^{t,r}, x)\Big|_{p=P^t(z)} = \frac{1}{P^t(z)} \left(\underbrace{\mu_{ts}(z,x)}_{\text{mean potential outcome}} - \underbrace{m_{ts}(P^t(z), z^{t,r}, x)}_{\text{mean observed outcome}} \right).$$

The logic is as follows. When $P^t(z)$ increases while $z^{t,r}$ is held fixed, the comparison between treatments t and r remains unchanged, while treatment s becomes strictly less attractive relative to both t and r. This is because an increase in $P^t(z)$ with $z^{t,r}$ fixed must come from a decrease in $z^{s,r}$. This decrease contracts treatment s along the (s,r) and (t,s) margins. However, because the selected-outcome regression conditions on $T_i = t$, only movements into or out of treatment t affect m_{ts}. The partial derivative of m_{ts} in the propensity argument therefore reflects only the movement of cases into treatment t across the (t,s) margin. Multiplying this derivative by $P^t(z)$ and adding the mean observed outcome gives the mean potential outcome under treatment t for cases at the (t,s) margin.

Applied to every pair (t,s), the preceding argument nonparametrically identifies the mean potential outcome under either treatment at that margin. The resulting identifying equations are collected in the following proposition.

Proposition 1 (Margin-Specific Mean Potential Outcomes)**.** *Suppose Assumptions 1 and 2 hold and $m_{ts}(p,a,x)$ is continuously differentiable in p. Let $x \in \text{supp}(X_i)$ be fixed. Assume that $\mathbb{E}[Y_i \,|\, X_i = x, T_i = t, Z_i = z]$ for every $t \in \{0,1,2\}$ and $P(z)$ are identified for all z in an open subset of the plane. Then for any $t, s \in \{0,1,2\}$ with $t \neq s$ and z in that open subset, it holds that*

$$\mu_{ts}(z,x) = \left[m_{ts}(p,a,x) + p\partial_p m_{ts}(p,a,x) \right]_{p=P^t(z), a=z^{t,r}}$$

with $r = r(t,s)$ denoting the remaining treatment in $\{0,1,2\} \setminus \{t,s\}$.

Replacing $Y_i(t)$ by $\mathbb{1}\{Y_i(t) \leq y\}$ identifies the distribution of $Y_i(t)$ at the margin, and hence any functional of that distribution, such as quantiles. Of the two conditioning variables, the derivative is taken in the treatment-t propensity while the relative index $z^{t,r}$ is held fixed as a control variable. Holding $z^{t,r}$ fixed ensures that treatment t draws cases from treatment s alone, not from the remaining treatment r. This property of our instruments is the key to identifying mean potential outcomes and treatment effects at each margin. By Proposition 1, each margin-specific treatment effect function can be identified as

$$\begin{aligned}\tau_{ts}(z,x) = \Big(m_{ts}(P^t(z), z^{t,r}, x) - m_{st}(P^s(z), z^{s,r}, x)\Big) \\ + P^t(z)\partial_p m_{ts}(P^t(z), z^{t,r}, x) - P^s(z)\partial_p m_{st}(P^s(z), z^{s,r}, x). \qquad (8)\end{aligned}$$

We close this subsection by comparing the identification strategy with the two approaches available for margin-specific effects under multiple unordered treatments, those of Lee and Salanié (2018) and Mountjoy (2022). Both identify their parameters through a ratio of derivatives, an outcome-by-treatment regression in the numerator and a treatment propensity in the denominator, whereas Proposition 1 differentiates the selected-outcome regression alone. Table **??** summarizes the comparison.

Lee and Salanié (2018) identify the mean potential outcome of cases for which the examiner assesses all three treatments equally. The parameter is the ratio of the cross-partial derivatives of the outcome-by-treatment regression $\mathbb{E}[Y_i T_i^t \,|\, X_i = x, Z_i = z]$ and of the propensity $P^t(z)$ in the two indices. It is finer than ours and costs an additional order of differentiation of both objects. In Proposition F.1 of the Appendix, we identify it from a derivative of μ_{ts} with respect to a derivative of the propensity.

Mountjoy (2022) identifies the same parameter as Proposition 1. A marginal increase in z^t draws cases into treatment t from both remaining treatments and moves no case between them, and the mean potential outcome under treatment s at the (t,s) margin is identified by the ratio $\partial_{z^t}\mathbb{E}[Y_i T_i^s \,|\, X_i = x, Z_i = z]/\partial_{z^t} P^s(z)$. The numerator and the denominator should typically be estimated separately, and the estimated ratio can be unstable where $\partial_{z^t} P^s(z)$ is small relative to the sampling error of its estimate. Our approach dispenses with the ratio and sidesteps such instability.

2.3 The Outcome Specification in the Second Stage

The identification results of Section 2.2 motivate a selected-outcome-regression approach to estimating margin-specific treatment effects. Implementing this approach raises three main challenges. First, the propensity function $P_o(\cdot)$ and the index instrument Z_i are not directly observed and must be estimated in the first stage, as described in Section 4.1. Second, the derivative $\partial_p m_{ts}$ in Proposition 1 is nonparametrically identified only if Z_i varies continuously conditional on X_i, whereas Z_i takes finitely many values given X_i, one per examiner. This difficulty arises in designs that pursue marginal treatment effects with discrete instruments (Brinch et al., 2017), including instruments built from examiner assignment (Kamat et al., 2024). Third, even where identification is secured, fully nonparametric estimation is infeasible in practice because X_i is typically moderate- to high-dimensional. This subsection introduces a semiparametric specification for the second-stage regressions that addresses the second and third challenges.

Let the (t, s) margin be of interest, and let $r \in \{0, 1, 2\} \setminus \{t, s\}$ be the remaining treatment. The objective is to recover the margin-specific effect function $\tau_{ts}(\cdot)$ using (8). This requires estimation of the selected-outcome regressions $m_{ts}(\cdot)$ and $m_{st}(\cdot)$, together with their partial derivatives with respect to their propensity arguments. Estimating these objects in a fully nonparametric manner is often infeasible due to the curse of dimensionality, while imposing fully parametric restrictions may be too rigid and lead to substantial misspecification bias.

As a flexible yet tractable compromise, we adopt a semiparametric varying-coefficient specification for the selected-outcome regressions. Specifically, we model each regression as linear in X_i with coefficients varying flexibly with the relevant treatment propensity and the relative index:

$$\begin{aligned} m_{ts}(P_i^t, Z_i^{t,r}, X_i) &= X_i' \beta_{ts}(P_i^t, Z_i^{t,r}), \\ m_{st}(P_i^s, Z_i^{s,r}, X_i) &= X_i' \beta_{st}(P_i^s, Z_i^{s,r}). \end{aligned} \tag{9}$$

Here, $\beta_{ts}(\cdot, \cdot)$ and $\beta_{st}(\cdot, \cdot)$ are vector-valued functions of the same dimension as X_i. The coefficients vary with the raw pair $(P_i^t, Z_i^{t,r})$, whose variation across cases derives from random examiner assignment.

The partially linear specification of Carneiro and Lee (2009) restricts (9) further, letting only the coefficient on the constant vary with the arguments while the slopes on the remaining covariates are fixed. Under that restriction the term $p\partial_p m_{ts}$ is a function of the

index alone, common to all cases with the same index value, whereas the varying-coefficient specification lets it vary with X_i and the index together and so accommodates selection patterns that differ across observed characteristics. Our goal is to estimate margin-specific effects and their heterogeneity across observed characteristics, which the richer specification delivers at the cost of greater estimation complexity, and the more parsimonious form remains available where that heterogeneity is not of interest.

The specification helps resolve the identification problem posed by the discreteness of Z_i given X_i. The coefficient functions are identified under the rank condition that $\mathbb{E}[X_i X_i' \,|\, T_i = t, P_i^t, Z_i^{t,r}] = \mathbb{E}[X_i X_i' \,|\, Z_i]$ is positive definite. Since the coefficient functions are continuous, it suffices that the rank condition holds on a dense subset of the support of Z_i. Rich variation in the case characteristics given the instrument substitutes for continuous variation in the instrument given the characteristics, since the examiner indices vary continuously across cases through the characteristics.

Under this specification, we have

$$\partial_p m_{ts}(P_i^t, Z_i^{t,r}, X_i) = X_i' \partial_p \beta_{ts}(P_i^t, Z_i^{t,r}), \tag{10}$$

and similarly for $\partial_p m_{st}$.

3 Identification of the First Stage

This section establishes identification of the parameters in the first-stage treatment assignment model presented in Section 2.1. Because the same propensity function $P_o(\cdot)$ generates every examiner's propensities, inverting it places them in a common index space, where the between-examiner comparisons underlying identification take place.

3.1 The Semiparametric Trinomial Model

Let $W_i = (\mathbb{1}\{J_i = 1\}X_i', \ldots, \mathbb{1}\{J_i = J\}X_i')'$ be the Jd_X-vector of regressors obtained by interacting the examiner indicators $\mathbb{1}\{J_i = j\}$ with the case characteristics. Let $t \in \{1, 2\}$ be a non-base treatment and write $\boldsymbol{\gamma}^t = \mathrm{vec}(\gamma_1^t, \ldots, \gamma_J^t) \in \mathbb{R}^{\dim(W_i)}$ for the vectorization of the $d_X \times J$ matrix of examiner-specific parameters for treatment t. Then the assessment of treatment t underlying the observed treatment assignment $T_i = T_i(J_i)$ in (2) can be

expressed as

$$X_i'\gamma_{J_i}^t - U_i^t = \sum_{j=1}^{J} \mathbb{1}\{J_i = j\} X_i'\gamma_j^t - U_i^t = W_i'\boldsymbol{\gamma}^t - U_i^t.$$

Let $\boldsymbol{\gamma} = (\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2)$ be the $\dim(W_i) \times 2$ matrix whose columns are the vectors of first-stage parameters across the two non-base treatments. This allows us to write $(W_i'\boldsymbol{\gamma}^1, W_i'\boldsymbol{\gamma}^2) = \boldsymbol{\gamma}'W_i$ as the vector of the predictable components of the two assessments for the non-base treatments.

Let $\boldsymbol{\gamma}_o$ denote the true value of $\boldsymbol{\gamma}$, and recall from Section 2.1 that the density f of U_i belongs to $\mathcal{F}$. Let $\gamma_{oj} = (\gamma_{oj}^1, \gamma_{oj}^2)$ be the true parameter matrix for examiner j, and let $P_o^t(z) = P^t(z \,|\, f)$ be the true t-treatment propensity as a function of the examiner indices $z = (z^1, z^2)$. Since U_i is independent of W_i, the model implies that the observed treatment assignment follows a trinomial distribution

$$T_i \,|\, W_i \;\sim\; \text{Mult}(P_o^t(\boldsymbol{\gamma}_o'W_i))_{t\in\{0,1,2\}}. \tag{11}$$

The model pairs a finite-dimensional parameter $\boldsymbol{\gamma}_o$, our chief object of interest, with an infinite-dimensional nuisance $P_o = (P_o^1, P_o^2)$. The model is therefore semiparametric, and the first-stage estimation problem is to estimate $\boldsymbol{\gamma}_o$ and P_o from the observed data $\{(T_i, W_i)\}_{i=1}^N$ under the trinomial model (11).

3.1.1 The Parameter Space

The parameter space for $\boldsymbol{\gamma}$ is taken to be a compact subset of $\mathbb{R}^{\dim(W_i)\times 2}$. Let $\boldsymbol{e} \in \mathbb{R}^{\dim(W_i)}$ be the vector with ones in the J intercept coordinates of W_i, the coordinate $\mathbb{1}\{J_i = j\}$ that multiplies the constant in each examiner block j, and zeros elsewhere, so that $\boldsymbol{e}'W_i = \sum_{j=1}^J \mathbb{1}\{J_i = j\} = 1$. Accordingly, $\boldsymbol{e}'\boldsymbol{\gamma}^1/J$ and $\boldsymbol{e}'\boldsymbol{\gamma}^2/J$ are the unweighted examiner means of the intercepts $(\gamma_{j,d_X}^1)_{j=1}^J$ and $(\gamma_{j,d_X}^2)_{j=1}^J$, respectively. Define

$$\Gamma = \{\boldsymbol{\gamma} \in \mathbb{R}^{\dim(W_i)\times 2} : |||\boldsymbol{\gamma}||| = 1,\, \boldsymbol{e}'\boldsymbol{\gamma}^1 = \boldsymbol{e}'\boldsymbol{\gamma}^2 = 0\}.$$

Here, $|||A||| := (\text{tr}(A'A))^{1/2}$ is the Frobenius norm of a matrix A, and $\boldsymbol{e}'\boldsymbol{\gamma}^1 = \sum_{j=1}^J \gamma_{j,d_X}^1$ and $\boldsymbol{e}'\boldsymbol{\gamma}^2 = \sum_{j=1}^J \gamma_{j,d_X}^2$ by definition, so the location restriction centers the intercepts across examiners. The Frobenius norm and the unit scale are adopted for concreteness and without loss of generality. The implementation reads the scale on an examiner-whitened norm, as described in Section G.1 of the Appendix.

Location and scale normalizations are required because the model alone cannot resolve ambiguities in location and scale: translating the intercepts $(\gamma_{j,d_X})_{j=1}^J$ and U_i by the same amount and scaling $\boldsymbol{\gamma}$ and U_i by the same positive constant leave the treatment propensity unchanged. The restrictions $|||\boldsymbol{\gamma}||| = 1$ and $\boldsymbol{e}'\boldsymbol{\gamma}^1 = \boldsymbol{e}'\boldsymbol{\gamma}^2 = 0$ pin down the scale and location of the parameters and rule out these operations. Conversely, any $\boldsymbol{\gamma}$ can be rescaled and translated to satisfy these restrictions, so these normalizations do not remove any identifiable content of $\boldsymbol{\gamma}$. These normalizations pin down the location and scale of U_i relative to those of the indices, so $P_o(\cdot)$ is pinned down. Section G.1 of the Appendix compares it with an alternative normalization.

3.1.2 The Function Class

We now describe the nuisance function class $\mathcal{P}$ for the propensity function $P = (P^1, P^2)$.

Let $\mathcal{Z} = \text{supp}(\boldsymbol{\gamma}_o' W_i)$ denote the support of the index instrument $Z_i = \boldsymbol{\gamma}_o' W_i$ at the true value $\boldsymbol{\gamma}_o$. The set $\mathcal{Z}$ is where the indices fall, and hence where the data determine $P(\cdot)$. The class defined below instead takes as its domain a compact rectangle containing $\mathcal{Z}$, because the indices $\boldsymbol{\gamma}' W_i$ move with $\boldsymbol{\gamma}$ and must remain in that domain at every candidate value. The compact support of X_i in Assumption 4(a) and the compactness of Γ make such a rectangle available and make $\mathcal{Z}$ compact as well.

Let $\bar{\mathcal{Z}}$ be a compact rectangle containing $\mathcal{Z}$, which will be defined more precisely in the next section when we specify the sieve space. Let $C^1(\bar{\mathcal{Z}})$ denote the class of continuously differentiable functions $P : \bar{\mathcal{Z}} \to \mathbb{R}^2$. We define the function class $\mathcal{P} = \mathcal{P}_{M,\kappa}$ as follows: for a sufficiently small $\kappa > 0$ and a sufficiently large $M > 0$, $\mathcal{P}$ is the collection of $P = (P^1, P^2) \in C^1(\bar{\mathcal{Z}})$ that satisfy, at every $z \in \bar{\mathcal{Z}}$,

(i) (Overlap.) $P^t(z) \geq \kappa$ for every treatment $t \in \{0, 1, 2\}$.

(ii) (Smoothness.) $\partial_{z^1} P^1(z) \leq M$ and $\partial_{z^2} P^2(z) \leq M$.

(iii) (Cross-response equality.) $\partial_{z^2} P^1(z) = \partial_{z^1} P^2(z)$.

(iv) (Monotonicity.) $\partial_{z^t} P^t(z) \geq 0$ for $t \in \{1, 2\}$, and $\partial_{z^s} P^t(z) \leq 0$ for every $t \in \{0, 1, 2\}$ and $s \in \{1, 2\}$ with $t \neq s$.

Here, $DP(z)$ denotes the Jacobian matrix of $P(\cdot)$ at z, which by (iii) is symmetric, with diagonal entries $\partial_{z^1} P^1(z)$ and $\partial_{z^2} P^2(z)$ and common off-diagonal entry $\partial_{z^2} P^1(z) = \partial_{z^1} P^2(z)$. Each of the conditions (i)–(iv) is a property shared by every propensity induced

by a density $f \in \mathcal{F}$; in particular, the true propensity P_o belongs to $\mathcal{P}$.[4] The class carries these properties over to candidate functions, which allow the estimation problem to be posed without reference to f.

Condition (i) requires that the propensity of each treatment is uniformly bounded away from zero by a constant $\kappa > 0$, referred to as the overlap condition. This is necessary to ensure that the likelihood and its score are well defined since the logarithm and reciprocal of treatment propensities are involved. It is also a condition for identifying margin-specific treatment effects in the second stage, as it relies on each t-treatment group exhibiting the common support of Z_i.

Condition (ii) requires that P responds smoothly to the indices by bounding the diagonal entries of its Jacobian by M, which together with condition (iv) bounds the whole of $DP(z)$. Such smoothness conditions are technically required to ensure that $\mathcal{P}$ is compact so that nonparametric optimization over $\mathcal{P}$ is well-posed and that the likelihood function responds smoothly to changes in $\boldsymbol{\gamma}$.

A smoothness condition on the link function $P(\cdot)$ is essential, as in any two-index model, where the index coefficients are identified through the gradient of the link with respect to the indices. In a generic two-index model, however, only the span $\{a\boldsymbol{\gamma}^1 + b\boldsymbol{\gamma}^2 : a, b \in \mathbb{R}\}$ is identified, because index sufficiency is invariant to invertible linear transformations of the indices. Smoothness alone therefore cannot separate $\boldsymbol{\gamma}^1$ from $\boldsymbol{\gamma}^2$. What breaks this invariance here is condition (iii), the cross-response equality $\partial_{z^2} P^1 = \partial_{z^1} P^2$. The assessment model forces it through the Williams–Daly–Zachary representation, as shown below. A generic two-index model does not impose it, and this structurally implied restriction pins down $\boldsymbol{\gamma}^1$ and $\boldsymbol{\gamma}^2$ separately rather than only their span.

Williams–Daly–Zachary Representation. The cross-response equality ensures that the propensity function $P(\cdot)$ arises as a gradient of a scalar function, which we refer to as the *social surplus function* and denote by $G(\cdot)$. To be more precise, let the social surplus function be defined by

$$G(z \,|\, f) := \mathbb{E}_{U_i \sim f}[\max(0, z^1 - U_i^1, z^2 - U_i^2)], \qquad z \in \mathbb{R}^2.$$

This function represents the expected maximum assessment across the treatments at the index vector z. We suppress the argument f wherever it is held fixed, writing $P(\cdot)$ and

[4] For $f \in \mathcal{F}$, the induced propensity function is also injective (Lemma I.4 in the Appendix).

$G(\cdot)$. According to the Williams–Daly–Zachary theorem (McFadden, 1981), the propensity function is the gradient of the social surplus function (Lemma I.3 in the Appendix),

$$P(z) = \nabla G(z) = \left(\frac{\partial G(z)}{\partial z^1}, \frac{\partial G(z)}{\partial z^2} \right)' \qquad \textit{for every } \; z \in \mathbb{R}^2.$$

By the symmetry of mixed partial derivatives, the cross-responses of the two non-base treatment propensities, the response of each to the other treatment's index, therefore satisfy the equality

$$\partial_{z^2} P^1(z) = \frac{\partial^2 G(z)}{\partial z^2 \partial z^1} = \frac{\partial^2 G(z)}{\partial z^1 \partial z^2} = \partial_{z^1} P^2(z),$$

which is the third condition in the definition of $\mathcal{P}$. This cross-response equality is sometimes called *Slutsky symmetry*, by analogy with consumer theory: the propensity function is the gradient of the surplus function just as compensated demand is the gradient of the expenditure function. Its Jacobian therefore inherits the symmetry of a Hessian in the same way as the Slutsky matrix. We use this term in the sequel. This relationship allows us to represent the propensity function $P(\cdot)$ in terms of a single scalar function $G(\cdot)$, which is useful for deriving a suitable sieve space in the following section.

Condition (iv) states that each treatment propensity is weakly increasing in its own index and weakly decreasing in the indices of the other treatments. The base treatment carries no index because its assessment is normalized to zero, and (iv) accordingly requires P^0 to be weakly decreasing in both indices. This is the substitution pattern of assignment to the treatment with the highest assessment: raising the assessment of one treatment weakly lowers the propensity of every other. Condition (iv) also makes every admissible surplus function convex. Since $P(\cdot) = \nabla G(\cdot)$, convexity of $G(\cdot)$ amounts to the positive semidefiniteness of $DP(\cdot)$, and every surplus function is convex by construction. Together with (iii), condition (iv) therefore makes $DP(z)$ symmetric and diagonally dominant with a nonnegative diagonal, hence positive semidefinite at every z.

This positive semidefiniteness distinguishes $P(\cdot)$ from its orientation-reversed image $P(-\cdot)$, whose Jacobian $-DP(-\cdot)$ is negative semidefinite. Without it, the alias $(-\boldsymbol{\gamma}_o, P_o(-\cdot))$ would be observationally equivalent to $(\boldsymbol{\gamma}_o, P_o)$, leaving the sign of $\boldsymbol{\gamma}_o$ unidentified. Imposing it removes the alias and forces the within-examiner alignment to a positive scaling.

The true propensity $P_o(\cdot)$ satisfies every inequality in (i), (ii) and (iv) strictly, uniformly

over $\bar{\mathcal{Z}}$, by the regularity of the underlying density f. This ensures that $\mathcal{P}$ is a convex space containing P_o in its interior, a property used in theoretical analysis of the SMLE.

3.2 Identification

No regressor in the examiner-IV framework serves as a numeraire, and without one the two index vectors are identified only up to their span. We show that within-examiner and between-examiner variation identify the first-stage parameters $\boldsymbol{\gamma}_o = (\gamma_{oj})_{j=1}^J$ and $P_o(\cdot)$ from the observed treatment propensities under primitive conditions on observable objects.

Recall that $\mathcal{Z} = \text{supp}(\boldsymbol{\gamma}_o' W_i) = \cup_{j=1}^J \text{supp}(\gamma_{oj}' X_i)$ is the index support. We call $(\boldsymbol{\gamma}, P)$ a candidate pair if $\boldsymbol{\gamma} \in \Gamma$ and $P = (P^1, P^2)$ is a continuously differentiable function on $\mathbb{R}^2$ that satisfies conditions (iii) and (iv) in the definition of $\mathcal{P}$ at every $z \in \mathbb{R}^2$, where $P^0 = 1 - P^1 - P^2$. We say that $\boldsymbol{\gamma}_o$ and the restriction of P_o to $\mathcal{Z}$ are identified if every candidate pair $(\boldsymbol{\gamma}, P)$ satisfying

$$P_o^t(\gamma_{oj}' x) = P^t(\gamma_j' x) \quad \text{for all } t \in \{0,1,2\},\ x \in \text{supp}(X_i),\ j = 1, \ldots, J$$

has $\boldsymbol{\gamma} = \boldsymbol{\gamma}_o$ and $P = P_o$ on $\mathcal{Z}$. Propensity functions that agree on $\mathcal{Z}$ are indistinguishable when paired with $\boldsymbol{\gamma}_o$. The left-hand side of the equality is determined by the observed t-treatment propensity given covariates $X_i = x$ and examiner $J_i = j$ since

$$\mathbb{P}(T_i = t \,|\, X_i = x, J_i = j) = \mathbb{P}(T_i = t \,|\, Z_i = \gamma_{oj}' x) = P^t(\gamma_{oj}' x \,|\, f).$$

Absent restrictions on $P(\cdot)$, these equalities determine the coefficients only up to their span, since $\gamma_{oj} A$ and $P_o((A')^{-1}\cdot)$ satisfy them for every invertible A. Recall from Section 3.1 that $P_o(\cdot)$ is injective on $\mathbb{R}^2$.

We leverage two types of variation in the data to identify $(\boldsymbol{\gamma}_o, P_o)$: within-examiner variation across cases and between-examiner variation across examiners. The within-examiner variation looks at cases assigned to the same examiner and exploits the variation in their covariates to identify the examiner-specific coefficients up to scale and location normalizations. In contrast, the between-examiner variation compares the treatment propensities across different examiners to resolve the scale and location ambiguities. For example, if two examiners j_1 and j_2 exhibit the same propensity at covariate values x and $\tilde{x}$, respectively, then the injectivity of the propensity function $P_o(\cdot)$ in the examiner index reveals that $\gamma_{oj_1}' x = \gamma_{oj_2}' \tilde{x}$, an equation that ties the relative scale and location of γ_{oj_1} and

γ_{oj_2}.

Comparability between two examiners presupposes that the supports of their propensities overlap, and pooling information across the values of a discrete covariate within an examiner presupposes the same across those values. Within-examiner identification treats the components of X_i asymmetrically, which calls for the partition $X_i = (X'_{a,i}, X'_{b,i}, 1)'$, where $X_{a,i}$ consists of continuous covariates, and $X_{b,i}$ contains the discrete covariates taking finitely many values, which we refer to as *slice variables*. Partition γ_j conformably as $\gamma_j = (\gamma_{j,a}, \gamma_{j,b}, \gamma_{j,d_X})$. Since J_i is assumed independent of X_i, conditioning on $J_i = j$ does not alter the covariate support.

Slice values x_b and x'_b are adjacent in the examiner-j cell if the supports of the propensity function given $(X_b = x_b, J = j)$ and given $(X_b = x'_b, J = j)$ share at least one interior point. The slice-adjacency component $C_j(x_b)$ collects the slice values connected to x_b by a finite chain of adjacent points. Two examiners j_1 and j_2 are adjacent if the supports of the propensity function given $J = j_1$ and given $J = j_2$ share at least one interior point, and the examiner-adjacency graph has the examiners $\{1, \ldots, J\}$ as vertices and the adjacent pairs as edges. Both notions require an interior point rather than a single common value, since a shared propensity value is traced back to a shared index value by inverting the candidate propensity function, which is one-to-one only where its Jacobian is nonsingular.

We are now prepared to state the conditions under which $(\boldsymbol{\gamma}_o, P_o)$ is identified.

Assumption 3. *The following hold.*

(a) (Continuous variation.) *For each examiner j, there exists $x_b^{j,*} \in \operatorname{supp}(X_b)$ such that $\operatorname{supp}(X_a \,|\, X_b = x_b^{j,*})$ has a non-empty interior in $\mathbb{R}^{\dim(X_a)}$.*

(b) (Rank.) *For each examiner j, $\gamma_{oj,a} \in \mathbb{R}^{\dim(X_a)\times 2}$ has full column rank 2.*

(c) (Nonseparability.) *For each examiner j, there exist no linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$ such that $v_1' DP_o(z) v_2 = 0$ for all $z \in \operatorname{Int}(\operatorname{supp}(\gamma_{oj}' X))$.*

(d) (Slice connectivity.) *For each examiner j, the affine hull of $C_j(x_b^{j,*})$ spans $\mathbb{R}^{\dim(X_b)}$.*

(e) (Examiner connectivity.) *The examiner-adjacency graph is connected.*

Theorem 1. *Under Assumptions 1 and 3, $\boldsymbol{\gamma}_o$ is identified in Γ and the propensity function $P_o(\cdot)$ is identified as a function on $\mathcal{Z}$.*

Assumptions 3(a) and (b) supply the continuous variation that drives within-examiner identification, Assumption 3(d) the slice pooling that extends it to the discrete coefficients, and Assumption 3(e) the between-examiner comparisons that place every examiner on a common scale and location, with the normalizations embedded in Γ fixing the remaining factors. The within-examiner argument rests on the symmetry of the propensity Jacobian. For an examiner j with two continuous characteristics, the responses of the observed propensities $\pi(x) = P_o(\gamma_{oj}'x)$ to x_a factor into the Jacobian in the indices and the coefficient matrix, $D\pi(x) = DP_o(\gamma_{oj}'x)\gamma_{oj,a}'$. Any candidate (γ_j, P) that reproduces π satisfies $DP(\gamma_j'x) = D\pi(x)\gamma_{j,a}^{-T}$ with $\gamma_{j,a}^{-T} = (\gamma_{j,a}^{-1})'$, and the symmetry of the left-hand side gives an equation in $\gamma_{j,a}$ and the observed $D\pi$ alone

$$D\pi(x)\gamma_{j,a}^{-T} = \gamma_{j,a}^{-1}D\pi(x)' \qquad \text{for all } x \in \text{supp}(X),$$

which $\gamma_{oj,a}$ solves. The coefficient matrix is fixed while $D\pi(x)$ varies with x, and Assumption 3(c) ensures that the only solutions are the scalar multiples of $\gamma_{oj,a}$, which identifies the coefficients up to scale. Were P_o separable into two univariate functions along a pair of directions, rescaling the two directions separately would produce further solutions, which makes Assumption 3(c) essentially necessary as well. Section F.3 of the Appendix discusses each condition and how the conditions can be assessed from an estimate of the observed propensity function.

4 Estimation

4.1 Estimation in the First Stage

4.1.1 Sieve Maximum Likelihood Estimation

The conditional distribution (11) implies a log-likelihood of the form

$$l_i(\theta) = l(T_i \,|\, W_i, \theta) = \sum_{t\in\{0,1,2\}} T_i^t \log P^t(\boldsymbol{\gamma}'W_i), \qquad \theta = (\boldsymbol{\gamma}, P) \in \Theta = \Gamma \times \mathcal{P},$$

where Γ is the parameter space for $\boldsymbol{\gamma}$ and $\mathcal{P}$ is the class of admissible propensity functions, both of which are defined in Section 3.1. Every examiner handles many cases, and the coefficients γ_j are estimated from the cases assigned to examiner j, while the propensity function is estimated from all cases. We define the sieve maximum likelihood estimator

(SMLE) as the solution to the following problem:

$$\hat{\theta} = (\hat{\boldsymbol{\gamma}}, \hat{P}) = \underset{\theta \in \Theta_N}{\arg\max}\ \mathcal{L}_N(\theta), \qquad \Theta_N = \Gamma \times \mathcal{P}_N, \tag{12}$$

where $\mathcal{L}_N(\theta) = N^{-1} \sum_{i=1}^{N} l_i(\theta)$ is the empirical criterion, and $(\mathcal{P}_N)_{N=1}^{\infty}$ is a sequence of sieve spaces approximating the function class $\mathcal{P}$ with dimension increasing with the sample size N. We adopt a sieve approach to estimation because direct maximization over the infinite-dimensional class $\mathcal{P}$ is computationally infeasible, and the sieve replaces it with a finite-dimensional problem.

4.1.2 The Sieve Space

While there are various choices of the sieve that can be used to approximate $\mathcal{P}$, we focus on the *gradient* B-spline sieve for concreteness and its computational advantages. This section introduces the definition of the gradient B-spline basis and presents the resulting finite-dimensional optimization problem for computing the SMLE.

Let $q \geq 4$ be the order of a univariate B-spline, which ensures that the spline is C^{q-2}. Let $\bar{\mathcal{Z}} = [\underline{z}^1, \bar{z}^1] \times [\underline{z}^2, \bar{z}^2]$ be a compact rectangle containing $\cup_{\boldsymbol{\gamma} \in \Gamma} \text{supp}(\boldsymbol{\gamma}' W_i)$, referred to as the knot box.[5] For a chosen integer $K \geq 1$, let $\underline{z}^1 = \eta_0^1 < \eta_1^1 < \cdots < \eta_K^1 = \bar{z}^1$ and $\underline{z}^2 = \eta_0^2 < \eta_1^2 < \cdots < \eta_K^2 = \bar{z}^2$ be sequences of knots for the first and second dimensions, respectively. For each dimension, we construct the univariate order-q B-spline basis functions defined on the corresponding knot sequence as $b_K^1(z^1) = \{b_{K,k}^1(z^1)\}_{k=1}^{K+q-1}$ and $b_K^2(z^2) = \{b_{K,k}^2(z^2)\}_{k=1}^{K+q-1}$. Then we define the bivariate B-spline basis functions on $\bar{\mathcal{Z}}$ as the tensor product of the univariate bases:

$$b_K(z) = \text{vec}(\{b_{K,k_1}^1(z^1) b_{K,k_2}^2(z^2)\}_{k_1,k_2=1}^{K+q-1}) \in \mathbb{R}^{d_K},$$

where $d_K = (K+q-1)^2$ is the dimension of the resulting basis so obtained. We simply denote $b_K(\cdot) = (b_{K,1}(\cdot), \ldots, b_{K,d_K}(\cdot))'$ as the vector of the bivariate B-spline basis functions.

The B-spline basis functions above are not directly used to approximate the propensity function $P(\cdot)$ since the latter is a gradient of the social surplus function $G(\cdot)$. Instead, we

[5] Under Assumption 4(a), the set $\cup_{\boldsymbol{\gamma} \in \Gamma} \text{supp}(\boldsymbol{\gamma}' W_i)$ is compact as a continuous image of $\Gamma \times \text{supp}(W_i)$ under the mapping $(\boldsymbol{\gamma}, w) \mapsto \boldsymbol{\gamma}' w$, and contains the true index support $\mathcal{Z}$; unlike $\mathcal{Z}$, it does not depend on the unknown $\boldsymbol{\gamma}_o$ and can be computed from Γ and the data. The same mapping makes $\mathcal{Z}$ compact, as the image of $\{\boldsymbol{\gamma}_o\} \times \text{supp}(W_i)$. Since every realized index $\boldsymbol{\gamma}' W_i$ with $\boldsymbol{\gamma} \in \Gamma$ lies in $\bar{\mathcal{Z}}$, the criterion below is well-defined over all of Γ. Computationally it is enough that the indices visited at each iteration lie in $\bar{\mathcal{Z}}$ since empty cells do not contribute to the likelihood.

consider the collection of gradients of the B-spline series, which is given by

$$\mathrm{GradSpl}(K) = \left\{ \nabla b_K(\cdot)'\xi_K \in \mathcal{P} : \xi_K \in \mathbb{R}^{d_K} \right\}.$$

The gradient B-spline sieve $(\mathrm{GradSpl}(K))_{K\geq 1}$ forms a dense subset in $\mathcal{P}$ as the mesh of quasi-uniform knots becomes fine. The admissible coefficient space for $\xi_K = (\xi_{K,1}, \ldots, \xi_{K,d_K})'$ is the set

$$\Xi(K) = \left\{ \xi_K \in \mathbb{R}^{d_K} : \nabla b_K(\cdot)'\xi_K \in \mathcal{P} \right\}.$$

Membership in $\mathcal{P}$ is a condition at every point of $\bar{\mathcal{Z}}$, so $\Xi(K)$ is in principle cut out by infinitely many constraints, each linear in ξ_K at a given z and forming a half-space in $\mathbb{R}^{d_K}$. The estimator is computed by an iterative ascent on the criterion, described in the Appendix. The monotonicity constraints are imposed at the nodes of a finite grid over the index support, which reduces them to finitely many linear inequalities, and every iterate is projected onto the resulting polytope when an ascent step leaves it. The overlap margin κ and the Jacobian bound M are taken sufficiently small and large so that they are inactive at the solution.

Since by the properties of the B-spline $\nabla b_K(\cdot)'1 \equiv 0$ where 1 is a d_K-vector of ones, we may impose $\xi_{K,d_K} = 0$ without loss of generality. Thus the effective dimension of the sieve space is at most $d_K - 1$ when all knot cells are loaded. A coefficient whose basis function vanishes on $\mathrm{supp}(Z_i)$ does not enter the likelihood, so the dimension the data identify is the number of basis functions that do not vanish there, which is smaller than $d_K - 1$ whenever the knot box is larger than the support. Notably, both P^1 and P^2 are approximated via a common coefficient vector ξ_K since they are gradients of the same social surplus function.

Let $K_N \to \infty$ be a sequence of integers increasing with the sample size N at a suitable rate, and let $\mathcal{P}_N = \mathrm{GradSpl}(K_N)$ be the sieve space for estimation. Let $P_\xi(\cdot) = \nabla b_{K_N}(\cdot)'\xi$ denote the propensity function corresponding to a coefficient vector $\xi \in \Xi(K_N)$, with components $P_\xi^t(z) = \partial_{z^t} b_{K_N}(z)'\xi$ for $t \in \{1, 2\}$. Since $\bar{\mathcal{Z}} \supseteq \cup_{\boldsymbol{\gamma}\in\Gamma} \mathrm{supp}(\boldsymbol{\gamma}'W_i)$, the candidate index value $\boldsymbol{\gamma}'W_i$ lies in $\bar{\mathcal{Z}}$ for every $\boldsymbol{\gamma} \in \Gamma$. $P_\xi(\boldsymbol{\gamma}'W_i)$ is therefore well-defined throughout Γ. The semiparametric MLE problem (12) is then converted into

$$(\hat{\boldsymbol{\gamma}}, \hat{\xi}) = \underset{\boldsymbol{\gamma}\in\Gamma,\, \xi\in\Xi(K_N)}{\arg\max}\ \mathcal{L}_N(\boldsymbol{\gamma}, P_\xi), \tag{13}$$

which is a finite-dimensional optimization problem over $\Gamma \times \Xi(K_N)$.

4.2 Estimation in the Second Stage

The second stage estimates the margin-specific treatment effect functions $\tau_{ts}(z,x) = \mu_{ts}(z,x) - \mu_{st}(z,x)$, which Proposition 1 expresses through the margin-specific selected-outcome regressions. Let $t \in \{0,1,2\}$ and $s \in \{0,1,2\} \setminus \{t\}$ be given and let $r \in \{0,1,2\} \setminus \{t,s\}$ be the remaining treatment. Under the specification of Section 2.3, the observed outcome under $T_i = t$ satisfies

$$Y_i = X_i' \beta_{ts}(P_i^t, Z_i^{t,r}) + \varepsilon_i, \qquad \mathbb{E}[\varepsilon_i \,|\, T_i = t, Z_i, X_i] = 0, \tag{14}$$

with $\beta_{ts}(\cdot) = (\beta_{ts,1}, \ldots, \beta_{ts,d_X})'(\cdot)$ the vector of varying coefficients. We estimate (14) by least squares on the treatment-t subsample, with the regressors $(P_i^t, Z_i^{t,r})$ replaced by their first-stage estimates and the coefficient functions approximated by a sieve.

The constructed regressors are

$$S_i^{t,r}(\theta) = (P^t(Z_i(\boldsymbol{\gamma})), Z_i^{t,r}(\boldsymbol{\gamma})), \qquad \hat{S}_i^{t,r} = S_i^{t,r}(\hat{\theta}), \qquad S_i^{t,r} = S_i^{t,r}(\theta_o), \tag{15}$$

where $Z_i(\boldsymbol{\gamma}) = \boldsymbol{\gamma}' W_i$, $Z_i^{t,r}(\boldsymbol{\gamma}) = Z_i^t(\boldsymbol{\gamma}) - Z_i^r(\boldsymbol{\gamma})$ with the convention $Z_i^0(\boldsymbol{\gamma}) = 0$, and $P^t(\cdot)$ is the propensity function of the same parameter; write $\hat{Z}_i = Z_i(\hat{\boldsymbol{\gamma}})$, $\hat{P}_i^t = P_{\hat{\xi}}^t(\hat{Z}_i)$ and $\hat{Z}_i^{t,r}$ for the estimated index, propensity and relative index. For the sieve, let $\tilde{b}_{\tilde{K}}(\cdot) \in \mathbb{R}^{d_{\tilde{K}}}$, $d_{\tilde{K}} = (\tilde{K} + \tilde{q} - 1)^2$, be the tensor-product B-spline basis of order $\tilde{q}$ built on $\tilde{K}$ knot intervals along each of the supports of P_i^t and $Z_i^{t,r}$, and approximate $\beta_{ts,\iota}(\cdot)$ by $\tilde{b}_{\tilde{K}}(\cdot)'\delta_\iota$ for $\iota = 1, \ldots, d_X$. With $\tilde{K}_N \to \infty$ as $N \to \infty$, the sieve least squares estimator is

$$(\hat{\delta}_1^{\mathrm{ls}}, \ldots, \hat{\delta}_{d_X}^{\mathrm{ls}}) = \underset{\delta_1, \ldots, \delta_{d_X} \in \mathbb{R}^{d_{\tilde{K}_N}}}{\arg\min} \sum_{i=1}^N T_i^t \Big(Y_i - \sum_{\iota=1}^{d_X} X_{\iota,i} \tilde{b}_{\tilde{K}_N}(\hat{S}_i^{t,r})' \delta_\iota \Big)^2 .$$

The generated regressors carry a bias into the second stage. Write $\varphi_i = X_i \otimes \tilde{b}_{\tilde{K}_N}(S_i^{t,r})$ for the second-stage regressor at the true first-stage parameter, $\hat{\varphi}_i$ for its constructed counterpart, and $v_i = \hat{\varphi}_i - \varphi_i$ for the construction error, where $\otimes$ denotes the Kronecker product. Let δ stack the coefficient vectors $(\delta_1, \ldots, \delta_{d_X})$, so that the regression reads $Y_i = \varphi_i' \delta + \varepsilon_i$ with $\mathbb{E}[\varepsilon_i \,|\, T_i, X_i, J_i] = 0$ on the t-subsample, up to the sieve approximation error. Substituting $Y_i = \hat{\varphi}_i' \delta - v_i' \delta + \varepsilon_i$ into the normal equations gives the identity

$$\hat{\delta}^{\mathrm{ls}} - \delta = -\hat{\mathbb{Q}}_N^{-1} \frac{1}{N} \sum_{i=1}^N T_i^t \hat{\varphi}_i v_i' \delta + \hat{\mathbb{Q}}_N^{-1} \frac{1}{N} \sum_{i=1}^N T_i^t \hat{\varphi}_i \varepsilon_i, \tag{16}$$

where $\hat{\mathbb{Q}}_N = N^{-1}\sum_{i=1}^N T_i^t \hat{\varphi}_i \hat{\varphi}_i'$ is the sample Gram matrix of the constructed regressor. Since the first stage uses no outcome data, $\hat{\varphi}_i$ is a function of the treatments, covariates and examiner assignments alone, the second term has mean zero, and the bias runs through the first term. Splitting $\hat{\varphi}_i = \varphi_i + v_i$ separates it into two sums.

The first sum $N^{-1}\sum_{i=1}^N T_i^t v_i v_i' \delta$ is quadratic in the first-stage error to first order, and its expectation is $\Sigma_v \delta$ with

$$\Sigma_v = \frac{1}{N}\mathbb{E}\left[T_i^t (X_i \otimes D\tilde{b}_{\tilde{K}_N}(S_i^{t,r})\dot{S}_i)\,\mathrm{Avar}(\hat{\theta})(X_i \otimes D\tilde{b}_{\tilde{K}_N}(S_i^{t,r})\dot{S}_i)'\right], \tag{17}$$

up to a remainder of third order in $\hat{\theta} - \theta_o$, where $D\tilde{b}_{\tilde{K}_N}(\cdot)$ is the derivative of the basis, $\dot{S}_i = \partial S_i^{t,r}(\theta_o)/\partial\theta'$ is the Jacobian of the constructed regressor in the first-stage parameter, and $\mathrm{Avar}(\hat{\theta})$ is the asymptotic variance of the first-stage estimator $\hat{\theta} = (\hat{\boldsymbol{\gamma}}, \hat{\xi})$. Its contribution to (16) is

$$-\hat{\mathbb{Q}}_N^{-1}\Sigma_v \delta, \tag{18}$$

the classical errors-in-variables form. Since Σ_v and $\mathbb{E}[T_i^t \hat{\varphi}_i \hat{\varphi}_i'] - \Sigma_v$ are both positive semidefinite to first order, this component is a contraction of δ in the norm induced by $\mathbb{E}[T_i^t \hat{\varphi}_i \hat{\varphi}_i']$. The magnitude of Σ_v is inherited from the first-stage sampling variance, which is large where examiners decide few cases, and the contraction is correspondingly severe there. Section F.8 of the Appendix gives the expansion and the two bounds.

The second sum $N^{-1}\sum_{i=1}^N T_i^t \varphi_i v_i' \delta$ contains the correlation between a case's contribution to the first-stage estimate and its own constructed regressor, the own-observation bias of leniency designs (Goldsmith-Pinkham et al., 2026). Removing this correlation as those designs do, by evaluating each case at a fit that excludes it, is also possible here through cross-fitting, although this would increase the computation time. The second sum also contains a term from the curvature of the basis that is quadratic in the first-stage estimation error, and cross-fitting does not remove it. In the application of Section 6, we choose to leave the second sum in place. The Monte Carlo experiments in Section 7 suggest that its overall effect on those estimates may be inconsequential.

We estimate Σ_v by its plug-in $\hat{\Sigma}_v$, with $\dot{S}_i$ evaluated at $\hat{\theta}$ and $\mathrm{Avar}(\hat{\theta})$ replaced by a pseudo-inverse of the sample information of Remark 3. The spline block is inverted over the eigenvectors whose eigenvalue is at least a threshold tol_N, which vanishes with the sample size at a rate given in Section 5.2, and the index block over the tangent space of the scale normalization. Section G.3 of the Appendix provides the details. The expansion behind (17) requires $\tilde{K}_N\|\hat{S}_i - S_i\| \to 0$, and Theorem 2(ii) bounds $\hat{S}_i - S_i$ uniformly only away

from the boundary of the support of Z_i. We accordingly form $\hat{\Sigma}_v$ from the cases whose fitted indices lie in a compact set $\mathcal{Z}'$ of the index space. In the application, this is achieved by trimming the data to a compact set in the index space. The sample counterparts $\hat{\mathbb{Q}}_N$ and $\hat{\Sigma}_v$ need not satisfy $\hat{\mathbb{Q}}_N \geq \hat{\Sigma}_v$, and in a direction violating it the estimated bias would exceed the entire coefficient. We therefore impose the bound on the sample spectrum. Let $\hat{\mathbb{Q}}_N^{-1/2}$ be the pseudo-inverse square root and $\hat{\mathcal{M}} = \hat{\mathbb{Q}}_N^{-1/2}\hat{\Sigma}_v\hat{\mathbb{Q}}_N^{-1/2}$, whose eigenvalue in a whitened direction is the share of that direction's variation attributable to first-stage error, and let $\min\{\hat{\mathcal{M}}, \mathbf{I}\}$ cap those eigenvalues at one. The corrected coefficients are

$$\hat{\delta} = (\mathbf{I} + \hat{\mathbb{Q}}_N^{-1/2} \min\{\hat{\mathcal{M}}, \mathbf{I}\}\, \hat{\mathbb{Q}}_N^{1/2})\hat{\delta}^{\text{ls}}, \tag{19}$$

partitioned as $\hat{\delta} = (\hat{\delta}_1, \ldots, \hat{\delta}_{d_X})$ conformably with $\hat{\delta}^{\text{ls}}$. The resulting corrected estimator is given by $\hat{\beta}_{ts}(\cdot) = (\hat{\beta}_{ts,1}(\cdot), \ldots, \hat{\beta}_{ts,d_X})'(\cdot)$ with $\hat{\beta}_{ts,\iota}(\cdot) = \tilde{b}_{\tilde{K}_N}(\cdot)'\hat{\delta}_\iota$ and the partial derivative $\widehat{\partial_1\beta}_{ts,\iota}(\cdot) = (\partial_1\tilde{b}_{\tilde{K}_N}(\cdot))'\hat{\delta}_\iota$ with respect to the propensity argument. The Monte Carlo experiments of Section 7 show that the attenuation is the conspicuous component of the bias at the empirical calibration and document the accuracy of the correction. The regularity conditions under which the expansion above holds, and the orders of its remainders, are provided in the Appendix.

Finally, the margin-specific mean potential outcome function $\mu_{ts}(z, x)$ can be estimated by

$$\hat{\mu}_{ts}(z, x) = x'(\hat{\beta}_{ts}(\hat{P}^t(z), z^{t,r}) + \hat{P}^t(z)\widehat{\partial_1\beta}_{ts}(\hat{P}^t(z), z^{t,r})),$$

where $z^{t,r}$ is the value of the relative index corresponding to z. Repeat the same procedure for $\beta_{st}(\cdot)$ and $\mu_{st}(\cdot)$ with the roles of t and s reversed. This results in the margin-specific treatment effect function estimated by

$$\begin{aligned}\hat{\tau}_{ts}(z, x) = x'\Big(&\hat{\beta}_{ts}(\hat{P}^t(z), z^{t,r}) - \hat{\beta}_{st}(\hat{P}^s(z), z^{s,r})\Big) \\ &+ x'\Big(\hat{P}^t(z)\widehat{\partial_1\beta}_{ts}(\hat{P}^t(z), z^{t,r}) - \hat{P}^s(z)\widehat{\partial_1\beta}_{st}(\hat{P}^s(z), z^{s,r})\Big).\end{aligned}$$

Section 5 establishes the large-sample properties.

5 Asymptotic Theory

This section establishes the large-sample properties of the sieve maximum likelihood estimator and the asymptotic normality of the aggregate effects estimated in the second

stage. Throughout, the sample size N grows while the number of examiners J stays fixed, which parallels small-n-large-T panel asymptotics. The examiner-specific coefficients $(\gamma_{oj})_{j=1}^{J}$ then remain finite in number, which sets aside the incidental parameters problem, and each is estimated from a within-examiner subsample whose size grows with N. The propensity function $P_o(\cdot)$ is common to all examiners and is estimated by pooling those subsamples.

5.1 Asymptotic Properties of the SMLE in the First Stage

We now turn to the large-sample properties of the SMLE (13) for both $\boldsymbol{\gamma}_o$ and $P_o(\cdot)$. The propensity function $P_o(\cdot)$ is treated as an infinite-dimensional nuisance parameter estimated at a slower nonparametric rate, whereas the finite-dimensional component $\hat{\boldsymbol{\gamma}}$ converges at the parametric rate of $N^{-1/2}$. The first-order theory establishes the consistency of $\hat{\boldsymbol{\gamma}}$ and its crude rate of convergence, and the efficient-score argument then refines the rate to semiparametric efficiency.

The location ambiguity of the examiner-specific intercepts is resolved by the mean-zero normalization $\boldsymbol{e}'\boldsymbol{\gamma}^1 = \boldsymbol{e}'\boldsymbol{\gamma}^2 = 0$ built into Γ. The parameter retains all examiner-specific intercepts, subject to these two linear restrictions; the efficient information $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$ below is accordingly singular in the two intercept-mean directions and the radial direction, and its Moore–Penrose inverse acts on their orthogonal complement.

5.1.1 Consistency and the Rate of Convergence of the SMLE

Assumption 4 below presents the conditions necessary for the first-order theory in addition to the identification conditions in Assumption 3. For an integer $k \geq 0$ and $\alpha \in (0, 1)$, a function is of class $C^{k,\alpha}$ on a set if its derivatives up to order k are continuous there and those of order k are Hölder continuous with exponent α, and a boundary is of class $C^{k,\alpha}$ if it is locally the graph of such a function. For a real $k > 0$ we write C^k for the class of k-times continuously differentiable functions when k is an integer and for $C^{\lfloor k \rfloor, k-\lfloor k \rfloor}$ otherwise, so that a single index records both the number of derivatives and the Hölder exponent of the highest one.

Assumption 4. *The following hold.*

(a) (I.I.D. sampling and finite examiners.) $(T_i, X_i, J_i)_{i=1}^{N}$ *are i.i.d. draws from a common distribution* $\mathbb{P}$. *Moreover,* X_i *has compact support and* $\lambda_j = \mathbb{P}(J_i = j)$ *is strictly positive for each* $j = 1, \ldots, J$ *with* J *fixed.*

(b) (Smoothness and interior.) *$P_o(\cdot) \in C^{k_o}(\bar{\mathcal{Z}})$ for (a real) $k_o \geq 2$, and it lies in the interior of $\mathcal{P}$ in the sense that $P_o^t(z) > \kappa$ for every $t \in \{0,1,2\}$ and $\|DP_o(z)\| < M$ for all $z \in \bar{\mathcal{Z}}$.*

(c) (Spline order and quasi-uniform mesh.) *$q \geq \max(k_o + 1, 4)$ and there exist positive constants $c_1, c_2 > 0$ such that $c_1 K_N^{-1} \leq \eta_m^t - \eta_{m-1}^t \leq c_2 K_N^{-1}$ for all $m = 1, \ldots, K_N$ and $t \in \{1,2\}$.*

(d) (Covariate and index density.) *For each $x_b \in \operatorname{supp}(X_b)$, the distribution of X_a given $X_b = x_b$ admits a density $p_{X_a|X_b}(\cdot \,|\, x_b)$ with respect to the Lebesgue measure on $\mathbb{R}^{\dim(X_a)}$ that is continuous and positive on its support. Moreover, the index instrument $Z = \boldsymbol{\gamma}_o' W$ has a compact support $\mathcal{Z}$ with a dense interior and admits a continuous and positive density $p_Z(\cdot)$.*

(e) (Index separation.) *There exist $\epsilon > 0$ and $c_3 > 0$ such that $\mathbb{E}[\operatorname{Var}(v'W_i \,|\, \boldsymbol{\gamma}'W_i)] \geq c_3 \|v\|^2$ for every $\boldsymbol{\gamma}$ with $|||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| < \epsilon$ and every v orthogonal to the columns $\boldsymbol{\gamma}^1$ and $\boldsymbol{\gamma}^2$ of that $\boldsymbol{\gamma}$ and to $\boldsymbol{e}$.*

(f) (Density bound and connected support.) *There are a neighborhood $\{\boldsymbol{\gamma} : |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| < \epsilon\}$ of $\boldsymbol{\gamma}_o$ and constants $0 < \underline{p} < \bar{p} < \infty$ such that, for every $\boldsymbol{\gamma}$ in this neighborhood, the index $\boldsymbol{\gamma}'W$ admits a density $p_{\boldsymbol{\gamma}}$ satisfying $\underline{p} \leq p_{\boldsymbol{\gamma}}(z) \leq \bar{p}$ for all $z \in \operatorname{supp}(\boldsymbol{\gamma}'W)$. Every interior point z of $\mathcal{Z}$ satisfies $z = \gamma_{oj}' x$ for some examiner j and some $x = (x_a, x_b, 1)$ with $x_b \in \operatorname{supp}(X_b)$ and x_a an interior point of $\operatorname{supp}(X_a \,|\, X_b = x_b)$.*

The gradient B-spline basis built on the order-(q,q) tensor-product B-splines for the social surplus function is of order $(q-1, q)$ and $(q, q-1)$ for the two components of the propensity function, and the quasi-uniform mesh of Assumption 4(c) delivers the spline approximation error $K_N^{-k_o}$. The index density of Assumption 4(d) and the two-sided density bound of Assumption 4(f) keep the spline Gram matrix uniformly well-conditioned on the interior of $\mathcal{Z}$, which is a standard condition in nonparametric regression. Assumption 4(e) requires that no linear combination $v'W_i$ be approximable in mean square by a function of the index $\boldsymbol{\gamma}'W_i$ near $\boldsymbol{\gamma}_o$, except in the directions for which it vanishes identically, the index directions themselves and the constant direction $\boldsymbol{e}$. It plays the role of the index-separation condition embedded in the semiparametric efficiency bound for multiple-index regression models, where the bound is finite only when the span of the index vectors is well separated from its linear complement. Primitive sufficient conditions adapted to the examiner design are given in the Appendix.

The following theorem establishes the consistency of the sieve MLE. Let $\rho_N := \sqrt{K_N^2 \log N/N + K_N^{-k_o}}$, which represents the nonparametric convergence rate of $\hat{\theta} = (\hat{\boldsymbol{\gamma}}, P_{\hat{\xi}})$ in the following L^2-metric:

$$\mathsf{L}(\hat{\theta}, \theta_o) = \|P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot)\|_{L^2(p_W)} = \left(\mathbb{E} \sum_{t\in\{0,1,2\}} (P_{\hat{\xi}}^t(\hat{\boldsymbol{\gamma}}'W) - P_o^t(\boldsymbol{\gamma}_o'W))^2\right)^{1/2},$$

where the expectation is taken with respect to the distribution of W_i, denoted by p_W. This metric arises naturally from maximum likelihood estimation, since the Kullback–Leibler divergence between the true and fitted propensities expands quadratically in $\mathsf{L}(\hat{\theta}, \theta_o)$.

Theorem 2 (Consistency and Rates of SMLE)**.** *Suppose that $K_N \to \infty$ and $K_N^2 \log N = o(N)$ as $N \to \infty$.*

(i) Under Assumptions 1, 3 and 4(a)–(c), it holds that $\hat{\boldsymbol{\gamma}} \to_p \boldsymbol{\gamma}_o$ and $\mathsf{L}(\hat{\theta}, \theta_o) = O_{\mathrm{P}}(\rho_N) \to_p 0$ as $N \to \infty$.

(ii) If, in addition, Assumptions 4(d)–(f) hold, then $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| = O_{\mathrm{P}}(\rho_N)$. Moreover, on the interior regions $\mathcal{Z}_N := \{z \in \mathcal{Z} : \mathrm{dist}(z, \partial\mathcal{Z}) \geq CK_N^{-1}\}$, with C depending only on q and the mesh ratio, the estimated propensity function attains the uniform rate $\|P_{\hat{\xi}}^t - P_o^t\|_{L^\infty(\mathcal{Z}_N)} = O_{\mathrm{P}}(K_N\rho_N)$ for $t \in \{1, 2\}$.

The rate of $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||$ in Theorem 2(ii) is inherited from the nonparametric rate through a local curvature bound of $\mathsf{L}(\hat{\theta}, \theta_o)$: once Theorem 2(i) places $\hat{\boldsymbol{\gamma}}$ in a neighborhood of $\boldsymbol{\gamma}_o$, $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| \lesssim \mathsf{L}(\hat{\theta}, \theta_o)$ with probability tending to one. The index-separation Assumption 4(e) creates the curvature along directions orthogonal to the column space of $\boldsymbol{\gamma}_o$, and the density Assumption 4(d) together with the nonseparability Assumption 3(c) create it within. The rate is crude in that $\hat{\boldsymbol{\gamma}}$ is tied to the slower nonparametric rate of $P_{\hat{\xi}}$; the parametric $N^{-1/2}$ rate requires the efficient-score argument of the next section.

5.1.2 Semiparametric Efficiency for the Index Coefficients

We first define the notion of semiparametric efficiency for the index coefficients, then identify the bound it refers to.

Definition 2. A regular estimator $\hat{\boldsymbol{\gamma}} \in \Gamma$ is *semiparametrically efficient for* $\boldsymbol{\gamma}_o$ if its asymptotic covariance matrix is minimal, in the positive semidefinite order, among the asymptotic covariance matrices of all regular estimators of $\boldsymbol{\gamma}_o$ in the first-stage model.

The first-stage model is the law of (T, W) through $l(t \mid w, \theta)$, with the outcome law $Y \mid T, W$ excluded. When the outcome mean is left unrestricted, the outcome carries no information about $\boldsymbol{\gamma}$ and the bound of Proposition 2 is also the bound in the full model of (Y, T, W). Section F.7 of the Appendix discusses when the outcome can add information. The efficient information is $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} = \mathbb{E}[s^{\text{eff}}_{\boldsymbol{\gamma}}(s^{\text{eff}}_{\boldsymbol{\gamma}})']$, formed from the efficient score

$$s^{\text{eff}}_{\boldsymbol{\gamma}} = \Pi_{\boldsymbol{\gamma}_o}\Big(s_{\boldsymbol{\gamma}} - \Pi_{\mathcal{T}} s_{\boldsymbol{\gamma}}\Big),$$

where $s_{\boldsymbol{\gamma}}$ is the index score of $l(t \mid w, \theta)$ at θ_o, $\Pi_{\mathcal{T}}$ is the $L^2(\mathbb{P})$-projection onto the nuisance tangent space $\mathcal{T}$, the closed linear span in $L^2(\mathbb{P})$ of the scores of the propensity perturbations $P_o + t\nabla\delta G$ at fixed $\boldsymbol{\gamma}_o$ with δG ranging over the smooth functions on $\bar{\mathcal{Z}}$, and $\Pi_{\boldsymbol{\gamma}_o}$ the projection onto the tangent space $\mathcal{T}_{\boldsymbol{\gamma}_o} = \{\Delta\boldsymbol{\gamma} : \langle \Delta\boldsymbol{\gamma}, \boldsymbol{\gamma}_o\rangle = 0,\ \boldsymbol{e}'(\Delta\boldsymbol{\gamma})^1 = \boldsymbol{e}'(\Delta\boldsymbol{\gamma})^2 = 0\}$ at $\boldsymbol{\gamma}_o$ with $\langle a, b\rangle = \text{vec}(a)'\text{vec}(b)$. Specifically, $\Pi_{\boldsymbol{\gamma}_o} = \mathbf{I}_L - \text{vec}(\boldsymbol{\gamma}_o)\text{vec}(\boldsymbol{\gamma}_o)'$ is the orthogonal projection onto $\mathcal{T}_{\boldsymbol{\gamma}_o}$, where $\mathbf{I}_L$ is the orthogonal projection onto $L = \{\Delta\boldsymbol{\gamma} : \boldsymbol{e}'(\Delta\boldsymbol{\gamma})^1 = \boldsymbol{e}'(\Delta\boldsymbol{\gamma})^2 = 0\}$, removing from each column its between-examiner intercept mean, and $\text{vec}(\boldsymbol{\gamma}_o)\text{vec}(\boldsymbol{\gamma}_o)'$ removes the radial direction, using $|||\boldsymbol{\gamma}_o||| = 1$; the two projections commute since $\boldsymbol{\gamma}_o \in L$. The presence of this projection departs from the textbook setting. Because $\boldsymbol{\gamma}_o$ satisfies $\boldsymbol{e}'\boldsymbol{\gamma}_o^1 = \boldsymbol{e}'\boldsymbol{\gamma}_o^2 = 0$ and $|||\boldsymbol{\gamma}_o||| = 1$, $\Pi_{\boldsymbol{\gamma}_o}$ annihilates the two intercept-mean directions and the radial direction $\text{vec}(\boldsymbol{\gamma}_o)$. $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$ is therefore singular, and Lemma C.1 in the Appendix shows that it is positive definite on $\mathcal{T}_{\boldsymbol{\gamma}_o}$:

$$v' I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v > 0 \quad \text{for all } v \in \mathcal{T}_{\boldsymbol{\gamma}_o} \setminus \{0\}.$$

The efficiency bound is then read through the Moore–Penrose inverse on $\mathcal{T}_{\boldsymbol{\gamma}_o}$ rather than an ordinary inverse.

Proposition 2. *The asymptotic covariance matrix of any regular estimator of $\boldsymbol{\gamma}_o$ in the first-stage model is bounded below by $(I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+$ in the positive semidefinite order. A regular estimator is therefore semiparametrically efficient for $\boldsymbol{\gamma}_o$ in the sense of Definition 2 exactly when*

$$\sqrt{N}\left(\text{vec}(\hat{\boldsymbol{\gamma}}) - \text{vec}(\boldsymbol{\gamma}_o)\right) \to_d \mathcal{N}\left(0, (I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+\right).$$

Every regular influence function takes values in $\mathcal{T}_{\boldsymbol{\gamma}_o}$, and efficiency is therefore decided on that subspace. A regular influence function is a gradient, hence orthogonal to the nuisance tangent space, and it differs from the canonical gradient $(I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+ s^{\text{eff}}_{\boldsymbol{\gamma}}$ by a term orthogonal to it, which delivers the bound. Section F.4 of the Appendix gives the argument.

Theorem 3 establishes that the SMLE attains this bound through an asymptotic linear representation with influence function $(I^{\text{eff}}_{\gamma\gamma})^{+} s^{\text{eff}}_{\gamma}$ and variance $(I^{\text{eff}}_{\gamma\gamma})^{+}$. The major obstruction is to control an increasing number of bias terms while deriving the distributional approximation, and to that end we make the following additional assumptions.

Assumption 5. *The following hold.*

(a) (Sieve growth.) *As* $N \to \infty$, $N = O(K_N^{2k_o+2} \log N)$ *and* $K_N = o(N^{1/9}/\log N)$.

(b) (Least favorable regularity.) *For some* $\alpha \in (0,1)$, *the conditional mean* $m_W(z) := \mathbb{E}[W_i \,|\, Z_i = z]$ *and the index density* p_Z *are of class* $C^{1,\alpha}(\mathcal{Z})$, *and* $\mathcal{Z}$ *is connected with boundary* $\partial\mathcal{Z}$ *of class* $C^{2,\alpha}$.

Assumption 5(a) trades bias against variance. It constrains the sieve dimension K_N from both sides: K_N must grow fast enough that the approximation bias $K_N^{-k_o}$ is at most of the order $\sqrt{K_N^2 \log N/N}$ of the stochastic error, and slowly enough that the stochastic terms and the increasing number of bias terms stay controlled. With $a_N \ll b_N$ denoting $a_N = o(b_N)$ and $a_N \lesssim b_N$ denoting $a_N = O(b_N)$, the admissible range for the sieve dimension is $(N/\log N)^{1/(2k_o+2)} \lesssim K_N \ll N^{1/9}/\log N$, hence $k_o > 7/2$, and Assumption 4(c) then implies $q \geq 5$. The upper bound is set by the cells that $\partial\mathcal{Z}$ crosses, on which the likelihood restricts the fitted propensity only on the intersection with $\mathcal{Z}$, and recovering the propensity and its gradient there costs a constant diverging with the reciprocal of the covered fraction. Assumption 5(b) concerns the least favorable direction for the index coefficients, the nuisance perturbation whose score is closest to the index score in the Fisher metric. That direction solves an elliptic variational problem whose coefficients are built from P_o, p_Z and m_W, and Assumption 5(b) lets it inherit two derivatives and a Hölder modulus from these primitives, which the spline sieve then approximates at the rate the expansion requires. The approximation bias of the propensity enters the expansion of $\hat{\boldsymbol{\gamma}}$ at first order only through its product with the approximation error of that direction, and the product is $o(N^{-1/2})$ under Assumption 5(a).

Theorem 3. *Under Assumptions 1, 3 and 4–5, it holds that*

$$\sqrt{N}(\operatorname{vec}(\hat{\boldsymbol{\gamma}}) - \operatorname{vec}(\boldsymbol{\gamma}_o)) = \left(I^{\text{eff}}_{\gamma\gamma}\right)^{+} \frac{1}{\sqrt{N}} \sum_{i=1}^{N} s^{\text{eff}}_{\gamma}(T_i, W_i) + o_{\mathrm{P}}(1).$$

Consequently, $\hat{\boldsymbol{\gamma}}$ *attains the bound of Proposition 2 and is semiparametrically efficient for* $\boldsymbol{\gamma}_o$ *in the sense of Definition 2.*

The constraints defining $\Xi(K_N)$ do not bind in the limit on any compact subset of the interior of $\mathcal{Z}$, since Assumption 4(b) places $P_o(\cdot)$ in the interior of $\mathcal{P}$ and $P_{\hat{\xi}}(\cdot)$ is eventually uniformly close to it there, as shown in Section F.4 of the Appendix. The rate differs examiner by examiner, since examiner j's block of $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$ carries the assignment probability λ_j and the matching block of the asymptotic covariance carries λ_j^{-1}, which places examiner j's coefficients at the effective rate $(N\lambda_j)^{-1/2}$. See Remark 2 in the Appendix for further discussion. The asymptotic covariance matrix is estimated from the sample information as if the sieve propensity were parametrically specified. See Remark 3 in the Appendix for further details.

5.2 Inference on the Aggregate Effects

In the empirical analysis, we summarize each effect function by its average under a weight on the index space. We show in this section that such an average can be estimated at the parametric rate, although the estimator of the effect function itself converges at a slower rate. The effect function involves the derivative of the coefficient function in the propensity, and integration by parts against a weight that vanishes on the boundary of its support replaces this derivative with the derivative of the weight.

We fix the margin (t, s) with remaining treatment r, a covariate value $x_0 \in \mathbb{R}^{d_X}$, and a weight $\upsilon : \mathbb{R}^2 \to [0, \infty)$ with $\int_{\mathbb{R}^2} \upsilon(z)dz = 1$. In the theory, we treat x_0, $\upsilon(\cdot)$ and the trimming rectangle $\mathcal{Z}'$ of Section 4.2 as known, and in the application we hold them at their sample values. For an index value z, the second-stage regressor of the (t, s) regression and its estimate are

$$\mathsf{S}^t(z) := (P^t_o(z), z^{t,r}), \qquad \hat{\mathsf{S}}^t(z) := (P^t_{\hat{\xi}}(z), z^{t,r}),$$

so that $S^{t,r}_i = \mathsf{S}^t(Z_i)$, and we define $\mathsf{S}^s(z)$ and $\hat{\mathsf{S}}^s(z)$ in the same way. The full-support condition of Assumption 1(c) makes S^t one-to-one. Our estimand and its estimator are

$$\bar{\tau}_{ts,o} := \int_{\mathbb{R}^2} \upsilon(z)\tau_{ts}(z, x_0)dz, \qquad \bar{\tau}_{ts} := \int_{\mathbb{R}^2} \upsilon(z)\hat{\tau}_{ts}(z, x_0)dz,$$

where $\hat{\tau}_{ts}$ is the estimator of Section 4.2, which evaluates the fitted coefficient functions at $\hat{\mathsf{S}}^t(z)$ and $\hat{\mathsf{S}}^s(z)$. Since $\hat{\tau}_{ts}(z, x)$ is linear in x, the aggregate $\bar{\tau}_{ts}(\mathcal{X})$ that we report in Section 6.3 equals $\bar{\tau}_{ts}$ with x_0 the mean of X_i over the cell $\mathcal{X}$ and $\upsilon(\cdot)$ the weight of that section divided by its integral.

The asymptotic variance involves one function on the regressor plane for each regression of the margin. We write ∂_1 for the derivative in the propensity argument and let

$$\begin{aligned} \upsilon^t(\mathsf{S}^t(z)) &:= \frac{\upsilon(z)}{|\det D\mathsf{S}^t(z)|}, \\ \mathcal{R}^t(\mathsf{S}^t(z)) &:= -\frac{|\det D\mathsf{S}^t(z)|}{p_Z(z)} \left(\mathbb{E}[X_i X_i' \,|\, Z_i = z]\right)^{-1} x_0 \partial_1 \upsilon^t(\mathsf{S}^t(z)), \end{aligned} \tag{20}$$

for z in the support $\mathcal{Z}_\upsilon$ of υ, with both functions zero at the other points of the regressor plane. We define $\upsilon^s(\cdot)$ and $\mathcal{R}^s$ in the same way with s in place of t. For each of the two regressions, let $\mathcal{S}$ be a compact subset of the interior of the support of its regressor.

Assumption 6. *The following hold for the regressions of β_{ts} and β_{st}.*

(a) (Sampling, errors and design.) *Assumption 4(a) holds with Y_i added to each observation. For constants $\bar{\sigma}^2, \underline{c} > 0$, the errors of (14) satisfy $\mathbb{E}[\varepsilon_i^2 \,|\, T_i, Z_i, X_i] \leq \bar{\sigma}^2$, and $\mathbb{E}[X_i X_i' \,|\, S_i] \geq \underline{c} \mathbf{I}_{d_X}$ almost surely on $\{S_i \in \mathcal{S}\}$.*

(b) (Weight and trimming.) *The weight υ is of class $C^{2,1}$ on $\mathbb{R}^2$, and its support $\mathcal{Z}_\upsilon$ lies in $\operatorname{Int} \mathcal{Z}'$. The image of $\mathcal{Z}'$ under S^t lies in the interior of the region $\mathcal{S}$ of the regression of β_{ts}. The same holds with S^s and β_{st} in place of S^t and β_{ts}.*

(c) (Mesh and smoothness.) *The second-stage knots are quasi-uniform, and the mesh covers a fixed rectangle of the regressor plane that contains a neighborhood of the support of the regressor. For some $\chi \geq 3 + \alpha$, with α as in Assumption 5(b), the coefficient functions are componentwise of class C^χ on the support of the regressor, and the spline order satisfies $\tilde{q} \geq \chi + 1$. The propensity function satisfies $k_o \geq 3 + \alpha$, and on a neighborhood of $\mathcal{Z}_\upsilon$ the matrix function $z \mapsto \mathbb{E}[X_i X_i' \,|\, Z_i = z]$ is of class $C^{1,\alpha}$.*

(d) (Sieve growth.) *With $\nu_N := K_N \rho_N$, the threshold tol_N of Section 4.2 satisfies $\nu_N = o(\mathsf{tol}_N)$, and*

$$\sqrt{N}\left(\tilde{K}_N^{-(\chi+1+\alpha)} + \tilde{K}_N^{-(1+\alpha)} \rho_N + \tilde{K}_N^{1-\chi} \nu_N\right) + \tilde{K}_N^2 \nu_N \to 0.$$

Assumption 6(b) keeps the images of the trimming rectangle inside the regions $\mathcal{S}$ on which Assumption 6(a) bounds the design from below. In Assumption 6(d), the term $\sqrt{N}\tilde{K}_N^{-(\chi+1+\alpha)}$ is the sieve bias of the aggregate relative to $N^{-1/2}$. If we take $K_N \asymp N^{1/10}$, which Assumption 5(a) admits for $k_o \geq 4$, and set $\chi = 7/2$ and $\alpha = 1/2$, the conditions

hold for $\tilde{K}_N \asymp N^b$ with $b \in (1/10, 3/20)$, up to logarithmic factors. This range contains $b = 1/(2\chi + 2) = 1/9$, at which the squared sieve bias and the variance of $\hat{\beta}_{ts}$ are of the same order.

Theorem 4 (Asymptotic Normality of the Aggregate Effects)**.** *Suppose that* (14) *holds for* (t, s) *and for* (s, t)*, and that Assumptions 1–6 hold. Let* $\bar{\tau}_{ts}$ *be formed from the corrected coefficients* $\hat{\delta}$ *of the two regressions, and let* Λ_{ts} *be the regular functional of the first stage in* (E.8) *of the Appendix, with efficient influence function* $\tilde{\psi}_{\Lambda_{ts}}$ *and variance* $\sigma^2_{\Lambda_{ts}}$ *as in Theorem C.1. Then*

$$\sqrt{N}(\bar{\tau}_{ts} - \bar{\tau}_{ts,o}) = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \psi_{ts}(Y_i, T_i, W_i) + o_{\mathrm{P}}(1) \to_d \mathcal{N}(0, \sigma^2_{ts}),$$

where, with $\varepsilon_{t,i} := Y_i - X_i'\beta_{ts}(S_i^{t,r})$ *and* $\varepsilon_{s,i} := Y_i - X_i'\beta_{st}(S_i^{s,r})$*,*

$$\begin{aligned} \psi_{ts}(Y_i, T_i, W_i) &:= T_i^t X_i' \mathcal{R}^t(S_i^{t,r}) \varepsilon_{t,i} - T_i^s X_i' \mathcal{R}^s(S_i^{s,r}) \varepsilon_{s,i} + \tilde{\psi}_{\Lambda_{ts}}(T_i, W_i), \\ \sigma^2_{ts} &:= \mathbb{E}\left[T_i^t \left(X_i' \mathcal{R}^t(S_i^{t,r})\right)^2 \varepsilon^2_{t,i}\right] + \mathbb{E}\left[T_i^s \left(X_i' \mathcal{R}^s(S_i^{s,r})\right)^2 \varepsilon^2_{s,i}\right] + \sigma^2_{\Lambda_{ts}}. \end{aligned}$$

If $\sigma^2_{ts} > 0$*,* $\tilde{K}_N^3 N^{-1/2} \to 0$ *and* $\mathsf{tol}_N = o(K_N^{-2})$*, the plug-in* $\hat{\sigma}^2_{ts}$ *of* (E.9) *in the Appendix satisfies* $\hat{\sigma}^2_{ts} \to_p \sigma^2_{ts}$*, and* $\sqrt{N}(\bar{\tau}_{ts} - \bar{\tau}_{ts,o})/\hat{\sigma}_{ts} \to_d \mathcal{N}(0, 1)$*.*

The three terms of ψ_{ts} are uncorrelated, since the two outcome terms vanish on disjoint events and each has mean zero given (T_i, X_i, J_i). Neither the outcome terms nor their variance depends on the sieve dimension $\tilde{K}_N$. The first-stage term collects the error of $\hat{\theta}$ through two channels. One is the movement of the generated regressors $\hat{S}_i$ in the fit, and the other is the evaluation of the fitted functions at $\hat{\mathsf{S}}^t(z)$ and $\hat{\mathsf{S}}^s(z)$. Each channel is a regular functional of the first stage, and Theorem C.1 applies to their difference Λ_{ts}. In the application, we cluster the standard errors at the judge-by-year level.

6 Empirical Application

6.1 Data

We study felony cases in the Virginia Circuit Courts, the same setting and data source as Humphries et al. (2025) (henceforth HOSSvD). Each case is assigned to one of the three dispositions (dismissal, noncarceral conviction, and incarceration) and to a judge

whose assignment is as good as random within cells differentiated by circuit and time of disposition. Table 7 in the Appendix reports the correlations of the estimated judge-specific coefficients with case characteristics. We follow the sample construction and the outcome definitions of HOSSvD except for the minimum caseload required of each judge, and Table 5 in the Appendix shows that the summary statistics by disposition are virtually identical to theirs. The full sample has 256,475 cases disposed by 129 judges from 2000 to 2019, and Section H of the Appendix details its construction. The second stage uses the cases disposed through 2012 whose fitted indices $\hat{Z}_i^1$ and $\hat{Z}_i^2$ both lie between the 0.25 and 99.75 percentiles of their sample distributions. This leaves $\tilde{N} = 167{,}581$ cases for the second stage out of 169,161.

The covariate vector X_i collects female and black indicators, three offense-type indicators for violent, property, and drug charges, which are on if any charge of the corresponding type is present, and the indicator for prior conviction within the last five years of disposition. Three further covariates are the time since the last prior conviction, the number of charges, and the number of misdemeanor charges, which enter as continuous regressors. The time since the last prior conviction is measured in days, and the two counts of charges are treated as continuous. Theorem 1 uses derivatives of the observed propensities in the continuous characteristics, and for the two counts the data identify arc slopes instead, which approximate those derivatives when the propensity function is close to linear between adjacent counts. The Monte Carlo designs of Section 7 keep the two counts at their integer sample values. Including the intercept, X_i has dimension 10 and there are 20 parameters per judge in the first-stage model.

The outcome of interest is recidivism, which is measured three ways: a new felony charge, a new felony conviction, and a new carceral sentence. We track each over four follow-up windows: the first year, years two to four, and years five to seven after disposition, along with the cumulative seven-year window.

6.2 First-Stage Estimates

The first stage estimates the assessment model of Section 2.1, augmented with common time effects in the examiner indices:

$$Z_i^t = X_i'\gamma_{J_i}^t + X_i'\gamma_{\text{year}(i)}^t + \alpha_{\text{month}(i)}^t + \alpha_{\text{dow}(i)}^t, \qquad t \in \{1, 2\},$$

where $\gamma^t_{\text{year}(i)}$ includes both year fixed effects and year-specific slopes on the non-intercept covariates, and $\alpha^t_{\text{month}(i)}$ and $\alpha^t_{\text{dow}(i)}$ are month and day-of-week fixed effects for treatment t. These parameters are identified by imposing the sum-to-zero constraint on each block of fixed effects and on the year slopes. These year slopes and time fixed effects are common to all judges, and absorb temporal variation in the disposition process. The identification analysis of Section 3 carries over to this specification, with the common time terms only enlarging each judge's index support, as detailed in Section F.2.

We estimate the semiparametric model by spline sieve maximum likelihood of order $q = 5$, as required for the efficiency of Section 5.1. Other tuning parameters (the knot box and the meshes of the knots and of the constraint grid) and their robustness are relegated to the Appendix.

Judge-specific coefficients. Figure 1 compares the coefficients recovered by the semiparametric fit with those of a logit, which keeps the judge-specific slopes and fixes the propensity at the multinomial logit form (5). Each panel plots the kernel density, across the 129 judges, of the judge-specific coefficient on one covariate in the conviction index z^1 or the incarceration index z^2, overlaying the semiparametric and logit estimates. Since the two fits are identified up to their own normalizations, both are placed on a common gauge before overlaying: judge intercepts are centered across judges on each axis, and the slope block of each fit is rescaled to a common norm.

The figure carries two messages. First, the slopes disperse substantially across judges under either fit, and this is the variation that a specification with judge-specific intercepts alone excludes. For most covariates the cross-judge spread is comparable to the mean loading. Under the Holm adjustment, Wald tests of the null that a coefficient is common across judges reject at the five percent level for both intercepts and for thirteen of the eighteen slopes (Section H.2 of the Appendix). The pattern of signs across judges is the same under either fit, which suggests that the qualitative structure of judge heterogeneity does not depend on whether the distribution of the unobserved components is restricted.

Second, the two fits differ in dispersion more than in location. The cross-judge means differ by less than a quarter of a cross-judge standard deviation for every term except the female loading, which the semiparametric fit places higher on the conviction index and higher on the incarceration index by about one standard deviation each. The dispersions differ in no single direction. On the incarceration index the logit distribution is the wider of the two for seven of the ten terms, by between 14 and 57 percent, and the narrower

for the loadings on prior conviction, on the number of charges, and on time since the last conviction, where the semiparametric distribution carries a long right tail the logit does not reach. On the conviction index the two dispersions are within 16 percent of each other for every term. The two fits differ only in whether the distribution of the unobserved components is restricted, and the differences in dispersion trace to that restriction, though which fit is the more accurate cannot be settled by this comparison alone. They concern how the estimated coefficients are spread across judges rather than where they are centered.

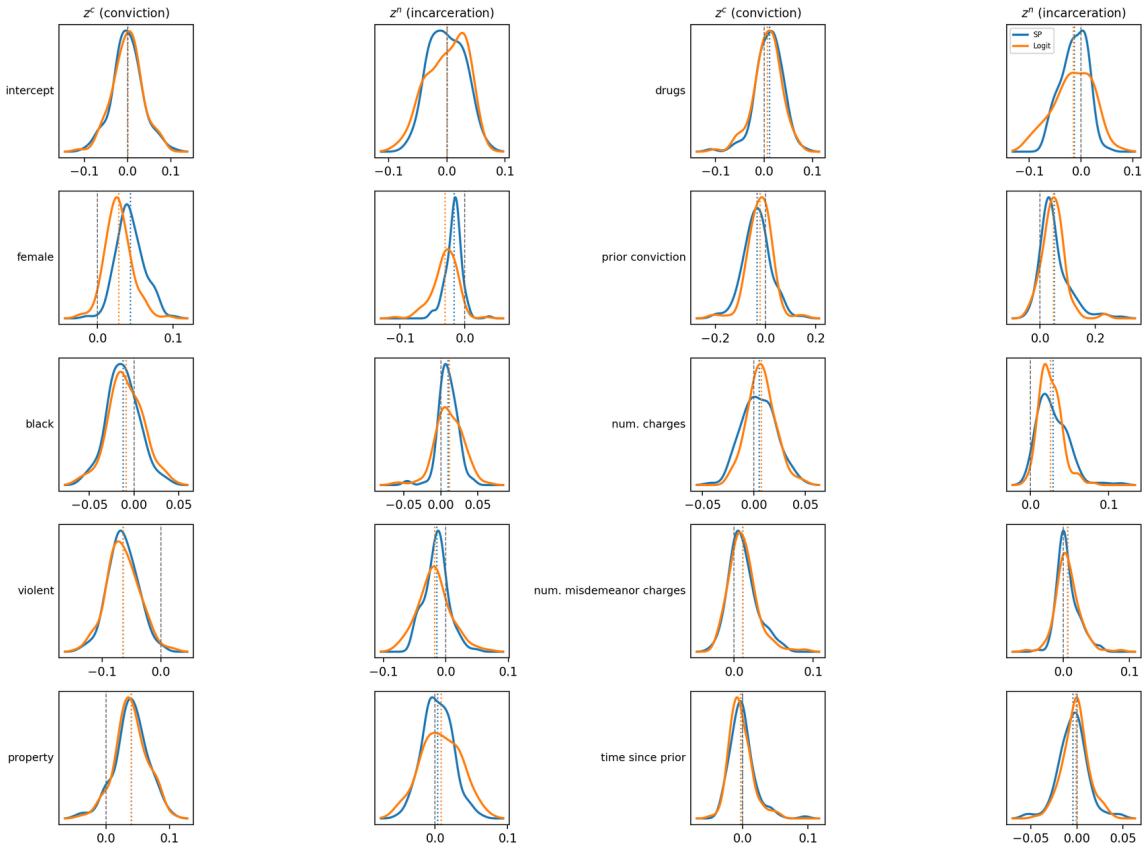
The semiparametric model attains the higher log-likelihood, at $-231{,}972$ against $-232{,}549$ for the logit. The gap of 576 units exceeds what the added flexibility of the sieve produces under a correct logit. The two fits differ only in the link, on which the sieve spends approximately 50 effective free coefficients, and in the Monte Carlo designs of Section 7, calibrated to this application with logistic errors, the realized gap is below 16.2 units for 95% of replications. The choice data therefore favor the unrestricted link, though the departure is small per observation, at 0.0022 units of log-likelihood. A small gap is, moreover, what a misspecified logit produces by construction. The logit pseudo-true value minimizes the Kullback–Leibler distance to the true choice probabilities, and with judge-specific intercepts and slopes the fitted probabilities can track the true ones closely under a wide range of misspecifications. What the pseudo-true fit cannot do is place the margins correctly, and it is the margins, not the fit, that the second stage uses. The likelihood comparison is accordingly a weak diagnostic for the constructed instruments, and the Monte Carlo experiments of Section 7 include a misspecified design in which the gap remains small per observation while the margins, and with them the estimated effects, move. Section H.1 of the Appendix describes the logit and the examiner-fixed-effects specification of HOSSvD in full and compares them with the semiparametric fit.

6.3 Margin-Specific Effects

Our main results use the second stage of Section 4.2, augmented with the circuit and time fixed effects to absorb cross-circuit and temporal variation not mediated by the index. The specification parallels the augmented first stage. For each margin (t, s), we estimate the outcome regression on the subsample with $T_i = t$,

$$Y_i = X_i'\beta_{ts}(S_i^{t,r}) + X_i'\beta_{\text{year}(i)} + \alpha_{\text{circ}(J_i)} + \alpha_{\text{month}(i)} + \alpha_{\text{dow}(i)} + \varepsilon_i, \quad \hat{S}_i^{t,r} = (\hat{P}_i^t, \hat{Z}_i^{t,r}),$$

Figure 1: Estimated Marginal Coefficient Distributions by First-Stage Specification



Notes. Each panel shows the kernel-smoothed density of the judge-specific coefficient across the judges on the labeled covariate in the conviction index z^1 or the incarceration index z^2, for the semiparametric and logit fits. The Gaussian kernel is used and the bandwidth is set to $\sigma J^{-1/5}$ with σ the sample standard deviation of each coefficient across the judges under each fit and $J = 129$. Continuous covariates (number of charges, number of misdemeanor charges, and time since last conviction) are standardized to have unit variance. The two fits are placed on a common gauge: intercepts are centered across judges within each axis, and each fit is rescaled so that the covariance-weighted Frobenius norm is the same. Dotted vertical lines mark the cross-judge means and the dashed line marks zero. The cross-judge spread reflects per-judge estimation noise in addition to judge heterogeneity.

where $\beta_{\text{year}(i)}$ includes year fixed effects and year-specific slopes on the non-intercept covariates, $\alpha_{\text{circ}(J_i)}$ is a circuit fixed effect corresponding to the assigned judge, and $\alpha_{\text{month}(i)}$ and $\alpha_{\text{dow}(i)}$ are the same time fixed effects as in the first stage. The index $\hat{Z}_i^t$ is the full index evaluated at the first-stage estimates, and the circuit and time controls account for the conditional randomization directly, as discussed in Section F.2. The augmented specification does not undermine consistency of the second stage established in Section D of the Appendix, with only finite-dimensional parameters added and not generated regressors. The margin-specific effect function $\tau_{ts}(\cdot)$ is then estimated from $\hat{\beta}_{ts}(\cdot)$ as described in Section 4.2. The coefficients on the circuit and time controls are estimated separately on each subsample $T_i = t$, and the effect function includes the difference between the fitted control terms of the two regressions, evaluated at the controls of the case. Our baseline specification approximates each coefficient function $\beta_{ts}(\cdot)$ by a tensor product of cubic B-splines with four basis functions per axis, the boundary knots at the minimum and maximum of the fitted indices in the trimmed sample and no internal knots. With no internal knots the basis spans the polynomials of degree three in each argument. The baseline coefficient function is therefore a bicubic polynomial in (p, a) with sixteen coefficients per covariate, and its derivative in the propensity is a polynomial of degree two in p. Section H of the Appendix compares this choice with a richer basis and with other alternatives.

We summarize the effect functions through their aggregates and covariate-cell averages, the estimands of interest. For $\mathcal{X} \subseteq \text{supp}(X_i)$, formed by either $\mathcal{X} = \text{supp}(X_i)$ or fixing one of the six binary case characteristics at zero or one, the aggregate effect on the (t, s) margin is the weighted average

$$\bar{\tau}_{ts}(\mathcal{X}) = \frac{\sum_{i=1}^{\tilde{N}} \mathbb{1}\{X_i \in \mathcal{X}\} \int_{\mathbb{R}^2} \upsilon(z) \hat{\tau}_{ts}(z, X_i) dz}{\sum_{i=1}^{\tilde{N}} \mathbb{1}\{X_i \in \mathcal{X}\} \int_{\mathbb{R}^2} \upsilon(z) dz},$$

where the weight is $\upsilon(z) = (1 - d(z)^2/d_q^2)^3$ for $d(z) \leq d_q$ and $\upsilon(z) = 0$ otherwise. Here, $d(z) = ((z - \bar{z})' \hat{V}_Z^{-1} (z - \bar{z}))^{1/2}$ is the Mahalanobis distance from the sample mean $\bar{z}$ of $(\hat{Z}_i)_{i=1}^{\tilde{N}}$, $\hat{V}_Z$ is the sample covariance of $(\hat{Z}_i)_{i=1}^{\tilde{N}}$, and d_q is the 0.875 sample quantile of $(d(\hat{Z}_i))_{i=1}^{\tilde{N}}$. The weight vanishes outside the interior region $\{z \in \mathbb{R}^2 : d(z) \leq d_q\}$.[6]

The aggregate figure reports each measure over the first year, years two to four, and years five to seven, together with the cumulative seven-year window. The measures are

[6]The interior of $\text{supp}(Z_i)$ is preserved under the open mapping $z \mapsto (P_o^t(z), z^{t,r})$, so the interior region is the same for every margin.

the new felony charge, the new felony conviction, and the new incarceration. A new incarceration follows a new conviction, and the difference between the effects on the two is the effect on new noncarceral convictions. Separating the noncarceral convictions from the carceral ones reveals into which type of reoffense each margin pushes the marginal case. The 95% confidence intervals are based on the normal approximation, the point estimate plus or minus 1.96 estimated standard errors. Table 6 in the Appendix reports the estimates and their standard errors.

The margins are labeled CD for noncarceral conviction relative to dismissal, NC for incarceration relative to noncarceral conviction, and ND for incarceration relative to dismissal. Figure 2 presents the aggregate effects. They average over the estimated index distribution, and the ND margin, which HOSSvD do not report, is estimated alongside the other two. Read together, the margins describe a revolving door: once the marginal case crosses into incarceration, its subsequent convictions arrive with carceral sentences.

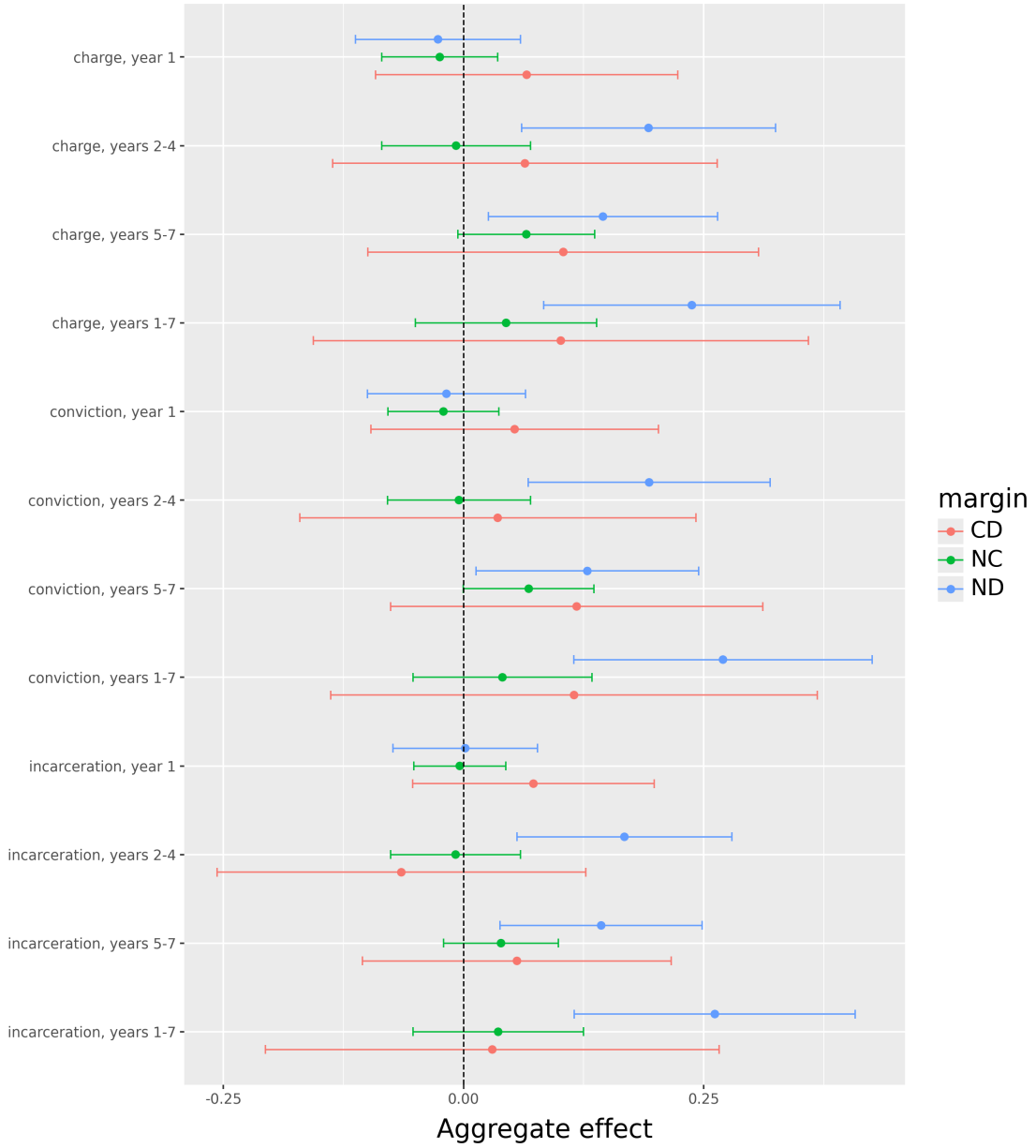
On the CD margin the point estimates are positive at every horizon except for new incarceration in years two to four. In the first year the three effects are of similar size, between 0.053 and 0.073. Over the seven-year window the effect on new convictions (0.115) is close to four times the effect on new incarceration (0.030).[7] The estimates suggest that a marginal conviction leads to more recidivism within seven years after disposition and that most of it is noncarceral, although the data do not separate any of these effects from zero.

On the NC margin every effect is at most 0.068 in absolute value. The effects on new charges and on new convictions are negative in the first four years, in the direction of incapacitation, though the intervals include zero, and positive in years five to seven, where they reach 0.065 and 0.068 and are significant at the ten percent level. The effect on new incarceration stays below 0.04 in absolute value, and its intervals cover zero at every horizon. Over the seven-year window the effects on new convictions (0.040) and on new incarceration (0.036) nearly coincide, which suggests a noncarceral component close to zero.

The ND margin carries the largest effects in the figure. The effects on new charges and on new incarceration are absent in the first year and accumulate thereafter, reaching 0.238 and 0.262 over the seven-year window. The effects on new convictions and on new incarceration differ by at most 0.026 at every horizon, which suggests that the excess

[7] Over years one to seven, Humphries et al. (2025, Table IV, column 8) report 0.235, 0.300 and 0.214 for new felony charges, convictions and incarceration, against our CD estimates of 0.101, 0.115 and 0.030 in Table 6.

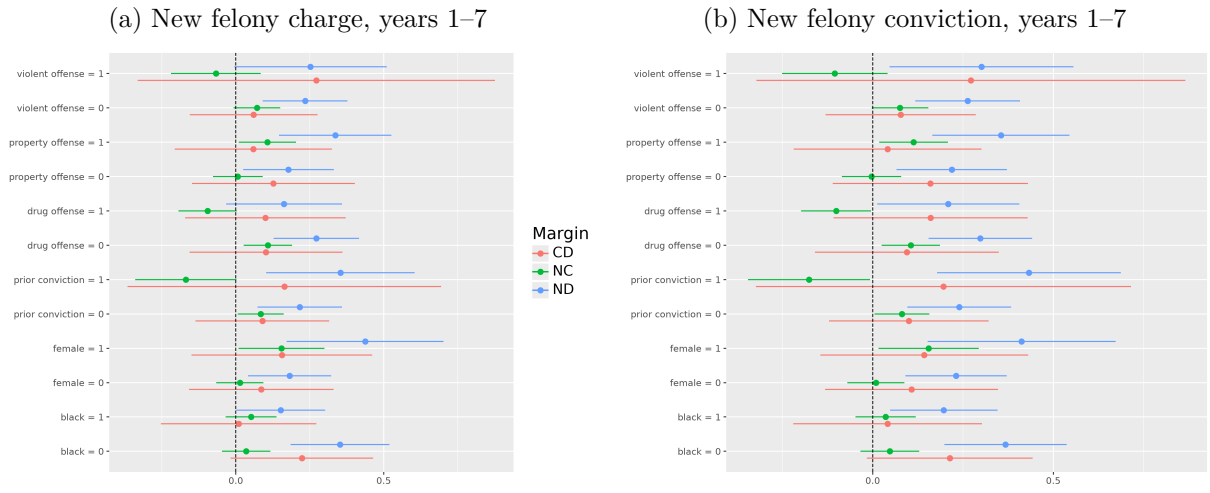
Figure 2: Aggregate Effects by Margin and Horizon



Notes. Each row reports the aggregate effect $\bar{\tau}_{ts}$ over the interior region for the indicated recidivism measure and horizon. Each bar represents the 95% interval described in the text.

recidivism on this margin is almost entirely carceral. The covariate-cell averages below show how this anatomy varies across observed characteristics.

Figure 3: Cell Average Effects by Margin: Cumulative Years 1 to 7



Notes. Each panel reports the cell average $\bar{\tau}_{ts}(\mathcal{X})$ for the indicated recidivism measure and window, with one row per covariate cell. The margins are labeled as in Figure 2. Each bar represents the 95% interval described in the text. Cells fix one binary characteristic at the indicated level and overlap across characteristics.

Figure 3 disaggregates the aggregate effects by covariate cell for new felony charges and new felony convictions over years one to seven. The panels for new incarceration and for years two to four are in Figure 5 of the Appendix. On the CD and ND margins the effects on new charges and on new convictions over years one to seven are positive in every cell, while the NC estimates vary in sign. Over both horizons the NC effects on all three measures are negative for cases with a violent charge, a drug charge or a prior conviction, and positive for cases without the respective characteristic. The effect on new charges is strongest for defendants with a prior conviction, reaching -0.171 over years two to four and -0.169 over years one to seven. This pattern is a descriptive regularity given the overlap of the cells rather than an effect of any one characteristic. The breakdown substantiates the revolving door. On the ND margin the effects on new convictions and on new incarceration differ by at most 0.114 in every cell over years one to seven, against effects on new incarceration between 0.192 and 0.443. On the NC margin the two effects differ by at most 0.039 in every cell but one. The exception is cases with a violent charge, where the effect on new incarceration (-0.227) is about twice the effect on new convictions

(−0.105). On the CD margin the effect on new convictions instead exceeds the effect on new incarceration in every cell, by 0.019 to 0.258 over years one to seven. On the ND margin the effects on new charges and on new incarceration are larger over years one to seven than over years two to four in every cell, most visibly for cases with a violent charge, where they more than double.

The confidence intervals are pointwise rather than adjusted for multiple testing. We therefore read the cell patterns through the signs and magnitudes of the estimates rather than through cell-by-cell significance.

7 Monte Carlo Experiments

The experiments serve two purposes. For the first stage, they substantiate the consistency and asymptotic normality results of Section 5.1 at the empirical calibration, and they measure the efficiency of the semiparametric estimator against a parametric benchmark that knows the error distribution. For the second stage, they assess the treatment-effect estimator at the empirical calibration, including the bias correction of Section 4.2, and they measure the coverage of the reported confidence intervals.

Both designs are calibrated to the empirical application in Section 6. The case characteristics X_i, the judge assignment J_i, and the circuit and time variables are held fixed at the first-stage sample of Section 6.1, retaining the judge caseload threshold and all other data cleaning conventions described in Section H of the Appendix, and the judge coefficients and common parameters are set to the empirical estimates of Section 6.2. In particular, every replication has exactly the sample size of the application, $N = 256{,}475$. Each replication redraws only the unobserved components U_i and regenerates the treatments by the argmax rule of (2), so every replication reproduces the sample's per-judge cell counts and respects the judge screens of Section 6.1. The two designs differ in the distribution of $U_i = (U_i^1, U_i^2)$. The first design, DGP 1, uses a location-scale logistic distribution. With $(\mathrm{ev}_i^0, \mathrm{ev}_i^1, \mathrm{ev}_i^2)$ independent standard type I extreme value variables as in (5), the standard logistic errors are $(\mathrm{ev}_i^0 - \mathrm{ev}_i^1, \mathrm{ev}_i^0 - \mathrm{ev}_i^2)$, and we set $U_i = \mu + \sigma(\mathrm{ev}_i^0 - \mathrm{ev}_i^1, \mathrm{ev}_i^0 - \mathrm{ev}_i^2)$. The location $\mu = (\mu^1, \mu^2) = (-0.15, -0.37)$ and the scale $\sigma = 0.3$ are chosen so that the average log-likelihood and the treatment shares approximately match the data. Under DGP 1 the logit is correctly specified, since the location and scale are absorbed by the judge intercepts and the normalization. The second design, DGP 2, sets $U_i \sim \mathcal{N}(\mu, \sigma^2 \mathbf{I}_2)$ with $\sigma = 0.28\pi/\sqrt{3}$, whose marginal variance is comparable to the first design's, and

$\mu = (0.02, -0.27)$ calibrated so that the aggregate treatment shares match the first-stage sample's. The logit propensity is misspecified under DGP 2. We have run 100 replications of each design.

The second stage simulates the outcomes using a reduced-form model of the selected-outcome regression, which is sufficient for the margin-specific effects. Let Z_i denote the true index fixed by the first-stage design and let Y_i be one of the twelve recidivism measures, the three measures of Section 6.1 over the four time windows. Given $T_i = t$, the outcome is generated by the varying-coefficient model

$$Y_i = X_i' \tilde{\beta}_t(Z_i) + X_i' \beta_{\text{year}(i)} + \alpha_{\text{circ}(J_i)} + \alpha_{\text{month}(i)} + \alpha_{\text{dow}(i)} + \varepsilon_i,$$

paralleling the specification of Section 6.3. Each component of $\tilde{\beta}_t(\cdot)$ is bilinear in the two indices, $\tilde{\beta}_{t,\iota}(z) = a_{\iota,0} + a_{\iota,1} z^1 + a_{\iota,2} z^2 + a_{\iota,12} z^1 z^2$. The fixed effects, the year slopes, and the coefficients $(a_{\iota,0}, a_{\iota,1}, a_{\iota,2}, a_{\iota,12})$ are set to ordinary least squares estimates on the outcome sample with the semiparametric indices in the role of Z_i, so the outcome model is also calibrated to the application. The reduced-form error ε_i is drawn independently across cases and independently of U_i from a normal distribution whose variance is set to the average squared residual of the corresponding calibrating regression. The calibrated effects roughly reproduce the sign pattern and the relative magnitudes of the empirical aggregates in Figure 2, and the design therefore evaluates the estimators at effect functions of realistic shape and magnitude. The first- and second-stage tuning parameters are all those of the empirical application. The estimand is the aggregate $\bar{\tau}_{ts}(\mathcal{X})$ of Section 6.3 with the true effect function τ_{ts} in place of $\hat{\tau}_{ts}$ and with the weight v formed from the estimated index of each replication. The estimand thus varies across replications only through the realized index, by amounts of order 10^{-3}, negligible at the scale of the effects.

In both designs we fit the two first stages of Section 6.2, the semiparametric estimator and the logit. Moving on to the second stage, we apply the varying-coefficient estimator of Section 4.2 with the errors-in-variables correction under each first stage, and compare both with an oracle benchmark that observes the true index and therefore requires no correction.

First stage. We report three summaries of the full-sample first-stage estimators. Table 1 compares the semiparametric and logit fits on the coverage rates of the 95% confidence intervals per coefficient, pooled across judges and replications, under both designs. Under

DGP 1 the empirical coverage lies close to the nominal level for nearly every coefficient and both fits. The ratios of the across-replication dispersion to the analytic standard errors concentrate around one, with medians of 1.00 for the semiparametric fit and 1.03 to 1.04 for the logit, and recentering every interval at its across-replication mean restores 0.95 coverage for both fits. The entries below the nominal level, most visibly the logit's coefficient on the number of charges on the incarceration axis, therefore reflect a small bias rather than a misquantified standard error.

Figure 4 displays, for the two judges of Figure 7, the sampling distributions of the semiparametric and logit estimates per coefficient across replications, with the truth, ± 1 SE and ± 2 SE boundaries marked, which speaks to the validity of the asymptotic normality established in Theorem 3. The figure also carries the efficiency comparison under DGP 1. The dispersions of the two fits are mostly comparable across judges, axes, and covariates, and the semiparametric estimator concedes little precision to the parametric benchmark that knows the error distribution. Across the judge-coefficient pairs, the ratio of the semiparametric to the logit standard error runs from 0.92 to 1.10 on the conviction axis and from 0.85 to 1.03 on the incarceration axis between its 5th and 95th percentiles.

Table 2 reports the L^2 norms of errors of the estimated indices $\hat{Z}^1$ and $\hat{Z}^2$ and of the relative index $\hat{Z}^1 - \hat{Z}^2$ under both designs, together with the maximized mean log-likelihood per observation of each fit. Both semiparametric and logit indices are well-centered so that the contribution of $N^{-1}\sum_{i=1}^{N}(\hat{Z}_i^t - Z_i^t) \approx 0$ to the L^2 error is negligible. The semiparametric fit recovers the index essentially as well as the correctly specified logit under DGP 1, its error slightly larger for $\hat{Z}^1$ and smaller for $\hat{Z}^2$ and the relative index. Under DGP 2 the semiparametric errors are as small or slightly smaller than under DGP 1, whereas the logit's grow throughout, most visibly for the relative index, whose error rises to 0.127 against 0.076 for the semiparametric fit. The relative index is an argument of the second stage on every margin. The deterioration under misspecification therefore reaches the estimated cell averages as shown below.

Judged by the fit alone, the two first stages differ little under either design. The mean log-likelihoods in Table 2 agree to the third decimal, and the gap in the maximized log-likelihood between the two fits, the benchmark cited in Section 6.2, has median 4.1 units and 95th percentile 16.2 under DGP 1 and median 35.9 and 95th percentile 50.5 under DGP 2, on the order of 10^{-5} and 10^{-4} per observation, respectively; DGP 2 therefore represents a mild misspecification of the logit rather than a gross one. The pseudo-true value of the logit approximates the choice probabilities closely under misspecification

of DGP 2, since the judge intercepts and slopes absorb much of the departure from the logistic link, but it does so with an index that places the margins differently from the truth. The index errors of Table 2 and the second-stage biases reported below are the consequence, while the likelihood shows minimal evidence of misspecification. The misspecification also manifests in the logit confidence intervals of Table 1. Averaged across the judge-coefficient pairs, the semiparametric coverage remains at 0.94 on both axes, while the logit coverage falls to 0.89 on the conviction axis and 0.83 on the incarceration axis, reaching a fifth percentile across pairs of 0.71 and 0.45 and a cross-judge average of 0.66 for the coefficient on the number of charges on the incarceration axis. Recentering every interval at its across-replication mean restores 0.95 coverage for both fits, which attributes the deterioration to misspecification bias rather than to the standard errors. The interval lengths remain comparable, with the ratio of the semiparametric to the logit standard error running from 0.90 to 1.21 on the conviction axis and from 0.89 to 1.22 on the incarceration axis between its 5th and 95th percentiles.

Table 1: Coverage of the First-Stage Confidence Intervals

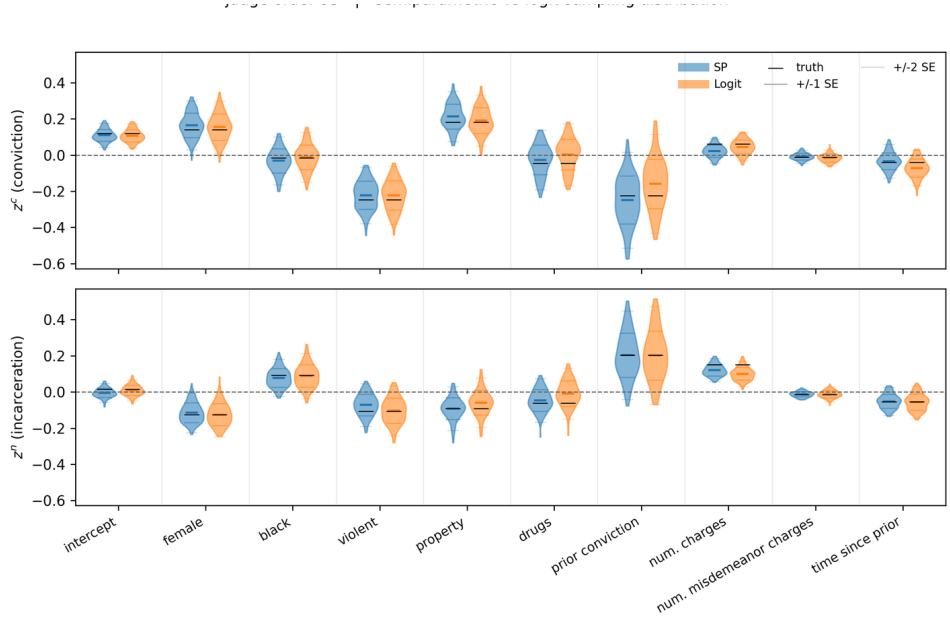
	DGP 1				DGP 2			
	Conviction (z^1)		Incar. (z^2)		Conviction (z^1)		Incar. (z^2)	
Covariate	Semi.	Logit	Semi.	Logit	Semi.	Logit	Semi.	Logit
Intercept	0.910	0.940	0.910	0.940	0.940	0.860	0.930	0.890
Female	0.940	0.950	0.940	0.940	0.940	0.910	0.930	0.860
Black	0.950	0.950	0.950	0.950	0.950	0.910	0.940	0.930
Any violent	0.940	0.950	0.920	0.930	0.940	0.930	0.950	0.810
Any property	0.950	0.930	0.940	0.950	0.940	0.900	0.950	0.890
Any drugs	0.950	0.930	0.940	0.890	0.950	0.930	0.940	0.840
Prior conviction	0.940	0.950	0.940	0.950	0.950	0.900	0.940	0.900
Num. charges	0.900	0.950	0.920	0.720	0.950	0.930	0.940	0.630
Num. misd. charges	0.940	0.950	0.950	0.950	0.950	0.940	0.950	0.940
Time since prior	0.950	0.930	0.940	0.950	0.950	0.930	0.950	0.940

Notes. Entries are coverage rates of the 95% confidence intervals per coefficient, averaged across judges and across the replications of each design, for the semiparametric and logit fits.

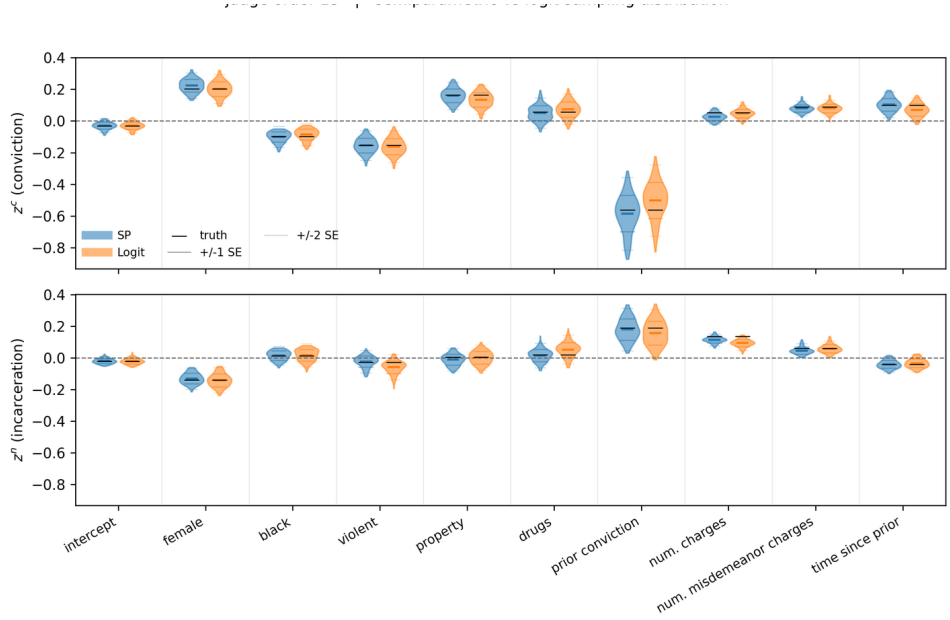
Second stage. We compare the two estimators over the cumulative window of years one to seven, by outcome, margin, and covariate cell. Table 3 reports the bias and the root mean squared error of each corrected estimator of the aggregate effects under both designs, together with the average length and the coverage rate of its 95% confidence

Figure 4: Sampling Distributions of Judge-Specific Coefficients under DGP 1

(a) Judge at the 20th caseload percentile



(b) Judge at the 80th caseload percentile



Notes. Each panel shows violin plots (Gaussian kernel densities with bandwidths selected by Scott's rule) of the sampling distributions of the semiparametric and logit estimates per coefficient across the 100 replications of DGP 1 for the two judges of Figure 7. The true coefficients are shown as black ticks, while ±1 SE and ±2 SE boundaries are marked by faint ticks.

Table 2: L^2 Errors of the Estimated Indices

		Z^1	Z^2	$Z^1 - Z^2$	Mean log-likelihood
DGP 1	Semiparametric	0.0887	0.0742	0.0730	−0.9063
	Logit	0.0854	0.0859	0.0756	−0.9062
DGP 2	Semiparametric	0.0739	0.0662	0.0750	−0.9119
	Logit	0.0917	0.1005	0.1262	−0.9120
L^2 norm of the truth		0.2519	0.2020	0.2748	

Notes. Entries are the L^2 errors of the estimated indices and of the relative index and the maximized mean log-likelihood per observation, averaged across the replications of each design, under the logistic design (DGP 1) and the normal design (DGP 2). The reported error is the L^2 norm of the index error over the sample, averaged across the replications, and the two fits are placed on the common gauge.

interval. Figures 8 and 9 of the Appendix display all four windows.

Under DGP 1 both estimators center close to the truth, and both attenuate the largest effects. The attenuation is most visible on the ND margin, which carries the largest true effects: the ratio of the mean estimate to the truth is 0.80 for new felony charges and 0.91 for new incarcerations with the semiparametric fit, against 0.91 and 0.98 with the logit. Table 2 suggests this magnitude, as the L^2 error of either fitted relative index is about one quarter of the L^2 norm of the truth, and measurement error of that relative size attenuates the slope of a linear regression by the factor $1/(1 + 0.27^2) \approx 0.93$. On the CD margin the semiparametric estimates lie above the truth, by 0.017 to 0.028. Across the entries of Panel A the root mean squared errors average 0.062 for the semiparametric fit and 0.058 for the logit, and the semiparametric estimator concedes little second-stage accuracy to the benchmark that knows the error distribution.

Under DGP 2 the semiparametric estimator remains close to the truth, with biases of at most 0.010 in absolute value, while the logit acquires a bias on the CD margin of −0.053 for new felony charges and −0.045 for new incarcerations. Across the entries of Panel B the root mean squared errors average 0.051 for the semiparametric fit and 0.057 for the logit, and the mean absolute bias is 0.005 against 0.020. The semiparametric estimator is also comparable to the oracle, whose biases reach 0.012 in absolute value and whose root mean squared errors average 0.042. The bias is not evenly distributed across cells, as Table 4 shows. For cases with a drug offense the bias sits on the NC margin, where

Table 3: Aggregate Effects across Replications: Semiparametric, Logit, and the Oracle

	Oracle		Semiparametric				Logit			
	Bias	RMSE	Bias	RMSE	Length	Coverage	Bias	RMSE	Length	Coverage
Panel A. DGP 1										
CD margin										
Chrg.	-0.005	0.066	0.025	0.091	0.356	0.97	-0.007	0.082	0.345	0.98
Conv.	0.014	0.073	0.028	0.097	0.348	0.93	0.009	0.091	0.336	0.92
Inc.	0.000	0.059	0.017	0.072	0.314	0.96	-0.008	0.072	0.304	0.97
NC margin										
Chrg.	0.001	0.029	0.006	0.046	0.157	0.88	-0.003	0.038	0.130	0.90
Conv.	0.005	0.026	0.011	0.036	0.154	0.95	0.005	0.030	0.128	0.97
Inc.	0.001	0.025	0.003	0.037	0.142	0.93	0.001	0.030	0.118	0.94
ND margin										
Chrg.	-0.008	0.044	-0.030	0.064	0.221	0.91	-0.013	0.060	0.233	0.93
Conv.	0.002	0.049	-0.026	0.060	0.218	0.93	-0.010	0.058	0.229	0.95
Inc.	0.003	0.041	-0.016	0.056	0.204	0.94	-0.003	0.058	0.214	0.95
Panel B. DGP 2										
CD margin										
Chrg.	0.001	0.041	-0.006	0.054	0.234	0.97	-0.053	0.090	0.318	0.94
Conv.	0.004	0.045	0.008	0.056	0.229	0.97	-0.027	0.081	0.310	0.97
Inc.	-0.003	0.037	-0.003	0.049	0.207	0.95	-0.045	0.082	0.282	0.90
NC margin										
Chrg.	0.004	0.043	0.006	0.049	0.185	0.94	0.002	0.036	0.138	0.95
Conv.	0.003	0.038	0.006	0.047	0.181	0.98	0.012	0.036	0.135	0.96
Inc.	-0.004	0.038	-0.002	0.041	0.168	0.95	-0.003	0.029	0.125	0.98
ND margin										
Chrg.	-0.012	0.046	-0.010	0.057	0.204	0.93	0.008	0.056	0.206	0.93
Conv.	0.003	0.047	0.008	0.056	0.201	0.94	0.016	0.059	0.203	0.91
Inc.	0.005	0.039	0.000	0.046	0.187	0.95	0.011	0.048	0.187	0.93

Notes. The semiparametric and logit entries refer to the corrected estimator and its 95% confidence interval. The oracle observes the true index and requires no correction. The intervals use standard errors clustered at the judge-by-year level. The outcomes are measured over years one to seven.

Table 4: Cell-Average Effects under DGP 2: Semiparametric, Logit, and the Oracle

	CD			NC			ND		
	Oracle	Semi.	Logit	Oracle	Semi.	Logit	Oracle	Semi.	Logit
Cases with a drug offense									
Chrg.	-0.001	-0.009	-0.028	0.001	0.007	0.041	-0.009	-0.008	-0.015
	(0.054)	(0.077)	(0.107)	(0.048)	(0.059)	(0.041)	(0.048)	(0.070)	(0.065)
Conv.	0.001	0.004	0.002	-0.001	0.009	0.052	0.002	0.000	-0.017
	(0.057)	(0.071)	(0.105)	(0.053)	(0.059)	(0.041)	(0.058)	(0.071)	(0.069)
Inc.	-0.002	-0.004	-0.012	-0.009	-0.002	0.027	0.005	0.000	-0.009
	(0.049)	(0.065)	(0.091)	(0.049)	(0.053)	(0.037)	(0.052)	(0.062)	(0.063)
Cases with a prior conviction									
Chrg.	0.009	0.011	0.076	0.007	0.019	0.018	-0.020	-0.010	-0.024
	(0.089)	(0.108)	(0.131)	(0.088)	(0.091)	(0.063)	(0.096)	(0.118)	(0.123)
Conv.	0.005	0.030	0.108	0.002	0.009	0.022	0.017	0.017	-0.010
	(0.087)	(0.104)	(0.128)	(0.080)	(0.090)	(0.067)	(0.091)	(0.104)	(0.103)
Inc.	-0.001	0.013	0.040	-0.002	0.006	-0.006	0.012	-0.008	-0.012
	(0.082)	(0.106)	(0.122)	(0.073)	(0.084)	(0.058)	(0.090)	(0.106)	(0.105)

Notes. Each entry is the bias of the oracle or of the corrected estimator, with the standard deviation in parentheses beneath. The outcomes are measured over years one to seven.

the logit estimate for new felony charges lies above the true effect of -0.17 by 0.041. For cases with a prior conviction the bias moves to the CD margin, where the mean logit estimate is 1.6 to 1.7 times the true effect for all three measures. Across both cells the semiparametric biases are comparable to the oracle's, at most 0.030 in absolute value against 0.020, while the logit's reach 0.108. A mildly misspecified first stage can thus distort the estimated effects for particular subpopulations by more than the aggregates reveal. Figures 10 and 11 of the Appendix display the two cells over all four windows. The remaining cell-average comparisons and the comparison without the bias correction are relegated to the Appendix.

Table 3 also turns from the point estimates to the intervals, and Table 9 in the Appendix reports the coverage rate and the average length of the semiparametric intervals over all four windows. The intervals are those of the corrected estimator, built as in Section G.4 with the loading that the correction of (19) induces. The outcome errors are drawn independently across cases in the designs above, while the standard errors are clustered at the judge-by-year level of the application. Forming them under independence instead leaves the picture unchanged, with average coverage higher by 0.1 percentage points in both designs.

Coverage stays near the nominal level. Over the thirty-six entries of Table 9, the

semiparametric coverage ranges from 0.88 to 0.99 under DGP 1 and from 0.90 to 0.98 under DGP 2, with averages of 0.94 and 0.95. The median ratio of the across-replication dispersion to the average standard error is 1.00 in both designs. Measured against the standard error of the uncorrected estimator, the corrected estimates disperse by a factor of 1.51 under DGP 1 and 1.45 under DGP 2 at the median entry. The logit intervals in Table 3 cover at rates between 0.90 and 0.98 in both designs. The semiparametric intervals are longer than the logit's on the NC margin, by about a fifth under DGP 1 and a third under DGP 2. Under DGP 2 the logit intervals are in turn about a third longer on the CD margin, where their coverage stays between 0.90 and 0.97 despite the bias.

8 Conclusion

Examiner designs do not directly provide instruments that shift cases toward one treatment without inducing selection into any other treatment. This paper constructs them from a multinomial model in which examiners choose the treatment they assess most favorably. Slutsky symmetry of treatment propensities helps identify the examiner-specific coefficients without exclusion restrictions on case characteristics, and the predicted assessments identify pairwise treatment effects at margins between two treatments. For felony cases in the Virginia Circuit Courts, the estimates suggest that incarceration raises reoffending relative to dismissal almost entirely through new carceral sentences. The effect of a noncarceral conviction relative to dismissal within seven years after disposition appears mostly noncarceral, though the data do not separate it from zero.

Formal specification testing and tests of Assumption 3 are left to future work. The linear index specification and the common distribution of the unobserved components across examiners restrict the observed propensities through index sufficiency in (4), which yields testable implications. With many characteristics and examiners, however, a test of these implications must account for the estimation error in the coefficients of all examiners, and the size of such a test has yet to be established. The rank and nonseparability conditions can be assessed informally from the first-stage estimates. The slice and examiner connectivity conditions can be inspected visually through the supports of the estimated propensities. These diagnostics are, however, not formal tests. Formal specification tests and formal counterparts of these diagnostics would help researchers examine whether the method is applicable to their own settings.

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Supplement to "Examiner and Judge Designs with Multiple Treatments"

This appendix collects the proofs of the results in the main text, together with the supporting lemmas and additional results.

Notation

We begin by fixing the notation used throughout the proofs.

Treatment propensities and the surplus. The propensity $P^t(z \mid f)$ and the social surplus function $G(z \mid f)$ are defined in the main text ((3) and Section 3.1), and (I.1) extends $P^t(z \mid f)$ to signed f. The Williams–Daly–Zachary representation $\partial G(z \mid f)/\partial z^t = P^t(z \mid f)$ is Lemma I.3, and explicit formulas for the derivatives $D_l P^t$ and $D_{ll'} P^t$ are in Lemma I.2.

Covering numbers. For a class $\mathcal{G}$ of functions and a semi-metric $d(\cdot, \cdot)$, $N(\epsilon, \mathcal{G}, d)$ is the smallest cardinality of an ϵ-net covering $\mathcal{G}$.

Differential operators. D is the Jacobian and ∇ is the gradient. The index is $z = (z^1, z^2) \in \mathbb{R}^2$. For a coordinate $t \in \{1, 2\}$, ∂_{z^t} is the partial derivative in that coordinate. We also write D_l for ∂_{z^l} and $D_{ll'}$ for $\partial_{z^l}\partial_{z^{l'}}$, and at times $D_1 = \partial_{z^1}$ and $D_2 = \partial_{z^2}$. The Hessian is D^2. For a propensity P^t, its gradient is $DP^t = \nabla P^t$ and its Hessian is $D^2 P^t$. The 2×2 Jacobian of the propensity function $P(z) = (P^1(z), P^2(z))$ is $DP = \nabla^2 G$, with entries $(\partial_{z^l} P^t(z))_{l,t \in \{1,2\}}$. For $P \in \mathcal{P}$ this Jacobian is symmetric and positive semidefinite (Lemma I.6), and for P_o it is positive definite (Lemma I.3).

Probability measures and stochastic order symbols. $\mathbb{P}$ and $\mathbb{E}$ are the population probability and expectation under the law of (Y_i, T_i, X_i, J_i). $\mathbb{P}_N$ and $\mathbb{E}_N$ are the empirical measure and expectation. We use them interchangeably, with $\mathbb{P}_N g = \mathbb{E}_N g = N^{-1} \sum_i g(Y_i, T_i, X_i, J_i)$. The empirical process is $\mathbb{G}_N = \sqrt{N}(\mathbb{P}_N - \mathbb{P})$. For a positive sequence a_N and random variables W_N, $W_N = O_{\mathrm{P}}(a_N)$ means W_N/a_N is tight, and $W_N = o_{\mathrm{P}}(a_N)$ means $W_N/a_N \to_p 0$.

B-spline surplus sieve. The tensor B-spline basis $b_K(z) \in \mathbb{R}^{d_K}$, the spline space $\mathrm{Spl}(K)$, the propensity sieve $\mathrm{GradSpl}(K) = \{\nabla b_K(\cdot)'\xi : \xi \in \Xi(K)\}$, and the constraint set $\Xi(K)$ enforcing the cap $P^t_\xi := \partial_{z^t} b_K' \xi \in [\kappa, 1 - \kappa]$ are defined in Section 4.1, with $d_K = (K + q - 1)^2 \asymp K^2$ and mesh width $h \asymp K^{-1}$. The proofs use the following further objects. The knot box $\bar{\mathcal{Z}}$ contains $\cup_{\boldsymbol{\gamma} \in \Gamma} \mathrm{supp}(\boldsymbol{\gamma}' W)$, and the cells Q are its mesh rectangles. The surplus potential with coefficient a is $s_a := b_K' a$, and $g^t := \partial_{z^t} b_K$, $g^0 := -(g^1 + g^2)$, so

$g^t(z)'a = \partial_{z^t} s_a(z)$. The set of charged cells is $\mathcal{A}$, and the charged region is $\Omega_{\mathcal{A}} := \cup_{Q\in\mathcal{A}} Q$. The charged seminorm is $\|a\|_{\mathcal{A}}^2 := \|\nabla s_a\|_{L^2(\Omega_{\mathcal{A}})}^2 = a' L_{\mathcal{A}} a$, and $L_{\mathcal{A}}$ is its Gram matrix.

Three pseudo-metrics. Fix parameters $\theta = (\boldsymbol{\gamma}, P)$ and $\tilde{\theta} = (\tilde{\boldsymbol{\gamma}}, \tilde{P})$. The Kullback–Leibler divergence $\mathrm{KL}(\theta\|\tilde{\theta})$, the Hellinger distance $\mathsf{H}(\theta, \tilde{\theta})$, and the L^2 (propensity) distance $\mathsf{L}(\theta, \tilde{\theta})$ are

$$\begin{aligned}
\mathrm{KL}(\theta\|\tilde{\theta}) &= \sum_{t\in\{0,1,2\}} \mathbb{E}\left[P^t(\boldsymbol{\gamma}'W) \log \frac{P^t(\boldsymbol{\gamma}'W)}{\tilde{P}^t(\tilde{\boldsymbol{\gamma}}'W)}\right], \\
\mathsf{H}(\theta, \tilde{\theta}) &= \left(\mathbb{E} \sum_{t\in\{0,1,2\}} \left(\sqrt{P^t(\boldsymbol{\gamma}'W)} - \sqrt{\tilde{P}^t(\tilde{\boldsymbol{\gamma}}'W)}\right)^2\right)^{1/2}, \\
\mathsf{L}(\theta, \tilde{\theta}) &= \left(\mathbb{E} \sum_{t\in\{0,1,2\}} (P^t(\boldsymbol{\gamma}'W) - \tilde{P}^t(\tilde{\boldsymbol{\gamma}}'W))^2\right)^{1/2}.
\end{aligned}$$

H and L are genuine metrics that are symmetric and satisfy the triangle inequality. In general, KL does neither.

Treatment-propensity classes. For $t \in \{1, 2\}$, the class of t-treatment propensities over the sieve is $\mathcal{G}_K^t := \{z \mapsto P_\xi^t(z) = \partial_{z^t} b_K(z)'\xi : \xi \in \Xi(K)\}$, and we collect the two as $\mathcal{G}_K := \mathcal{G}_K^1 \cup \mathcal{G}_K^2$. Each $\mathcal{G}_K^t$ lies in the linear space $\{\partial_{z^t} b_K(\cdot)'\xi : \xi \in \mathbb{R}^{d_K}\}$, of dimension at most d_K, and has constant envelope 1. It is therefore a VC-subgraph class of index at most $d_K + 2$.

Uniform entropy and the uniform entropy integral. For a class $\mathcal{F}$ with envelope F, $N(\epsilon, \mathcal{F}, \|\cdot\|)$ is the ϵ-covering number in $\|\cdot\|$. With the supremum taken over finitely supported probability measures Q, the uniform entropy and the uniform entropy integral are

$$\begin{aligned}
&\sup_Q \log N\left(\epsilon\|F\|_{L^2(Q)}, \mathcal{F}, L^2(Q)\right), \\
&\tilde{J}(\delta, \mathcal{F}) := \int_0^\delta \sqrt{1 + \sup_Q \log N\left(\epsilon\|F\|_{L^2(Q)}, \mathcal{F}, L^2(Q)\right)} d\epsilon.
\end{aligned}$$

These quantities enter the standard maximal inequality $\mathbb{E}\|\mathbb{G}_N\|_{\mathcal{F}} \lesssim \tilde{J}(1, \mathcal{F})\|F\|_{L^2(\mathbb{P})}$. It turns an entropy bound into a fluctuation bound. Here $\|\mathbb{G}_N\|_{\mathcal{F}} = \sup_{f\in\mathcal{F}} |\mathbb{G}_N(f)|$. This supremum is measurable when $\mathcal{F}$ is pointwise separable. The classes we use have this property.

A Identification

Proof of Proposition 1. Let (t, s) be a pair of treatments in $\{0, 1, 2\}$ with $t \neq s$, and let r be the remaining treatment. Let $x \in \text{supp}(X_i)$ be given. We will show that

$$\mu_{ts}(z, x) = [m_{ts}(p, a, x) + p\partial_p m_{ts}(p, a, x)]_{p=P^t(z), a=z^{t,r}}$$

holds for all z in the open subset of the plane in the hypothesis of Proposition 1, where $\mu_{ts}(\cdot)$ and $m_{ts}(\cdot)$ are functions defined in Section 2, and $P^t(\cdot)$ is the t-treatment propensity.

Step 1 (Reparametrization): Recall our assessment model: the assessments of 0, 1 and 2 are

$$0, \qquad z^1 - U_i^1, \qquad z^2 - U_i^2,$$

where $z = (z^1, z^2)$ is a vector of indices and $U_i = (U_i^1, U_i^2)$ is a vector of unobserved heterogeneity independent of (Z_i, X_i). The ordering of the assessments is invariant to a common shift. We reparametrize the model so that r becomes the base treatment with $\tilde{z}^r = 0$ and $\tilde{U}_i^r = 0$: subtracting $z^r - U_i^r$ from each of the three assessments gives the assessments of r, t and s,

$$0, \qquad z^{t,r} - U_i^{t,r}, \qquad z^{s,r} - U_i^{s,r},$$

where $z^{t,r}$ and $U_i^{t,r}$ are the differences in the indices and unobserved heterogeneity between treatments t and r, respectively, and analogous definitions hold for s. As noted, T_i is invariant to this reparametrization. Write $\tilde{z}^t = z^{t,r}$ and $\tilde{z}^s = z^{s,r}$, and similarly for $\tilde{U}_i^t$ and $\tilde{U}_i^s$. Then the original assessment model can be reparametrized with the assessments 0, $\tilde{z}^t - \tilde{U}_i^t$ and $\tilde{z}^s - \tilde{U}_i^s$ of r, t and s.

Step 2 (Mapping between the original and the reparametrized objects): Let $\tilde{P}^t(\cdot)$ be the t-treatment propensity under the reparametrization in Step 1, i.e., the propensity of treatment t induced by the distribution of $(\tilde{U}_i^t, \tilde{U}_i^s)$. Then as a function of z, the following identity holds:

$$\tilde{P}^t(\tilde{z}) = \tilde{P}^t(z^{t,r}, z^{s,r}) = P^t(z)$$

because the treatment assignment is invariant to the reparametrization and hence the resulting treatment propensities are equal. Furthermore, the reparametrization results in

the following margin between treatments t and s:

$$\{\tilde{U}_i \in \tilde{\mathcal{C}}_{ts}(\tilde{z})\} = \{\tilde{U}_i \in \tilde{\mathcal{C}}_{ts}(z^{t,r}, z^{s,r})\} = \{U_i \in \mathcal{C}_{ts}(z)\}$$

where $\mathcal{C}_{ts}(z)$ is the margin between treatments t and s defined in Section 2, and $\tilde{\mathcal{C}}_{ts}(\tilde{z}) := \{\tilde{u} \in \mathbb{R}^2 : \tilde{z}^t - \tilde{u}^t = \tilde{z}^s - \tilde{u}^s > 0\}$ is the corresponding margin under the reparametrization. This allows us to express the function $\mu_{ts}(\cdot)$ in terms of the mean potential outcome on the reparametrized margin $\tilde{\mathcal{C}}_{ts}(\cdot)$:

$$\tilde{\mu}_{ts}(\tilde{z}, x) = \tilde{\mu}_{ts}(z^{t,r}, z^{s,r}, x) = \mu_{ts}(z, x)$$

where $\tilde{\mu}_{ts}(\tilde{z}, x) := \mathbb{E}[Y_i(t) \,|\, X_i = x, \tilde{U}_i \in \tilde{\mathcal{C}}_{ts}(\tilde{z})]$.

Lastly, we observe that the margin-specific regression function $m_{ts}(\cdot)$ can be expressed as

$$\begin{aligned} m_{ts}(P^t(z), z^{t,r}, x) &\equiv \mathbb{E}[Y_i \,|\, X_i = x, T_i = t, P_i^t = P^t(z), Z_i^{t,r} = z^{t,r}] \\ &= \mathbb{E}[Y_i \,|\, X_i = x, T_i = t, Z_i = z], \end{aligned}$$

because the event $\{T_i = t, P_i^t = P^t(z), Z_i^{t,r} = z^{t,r}, X_i = x\}$ is equivalent to the event $\{T_i = t, Z_i = z, X_i = x\}$ by the one-to-one mapping between z and $(P^t(z), z^{t,r})$.[8] The mapping is one-to-one because, in the coordinates of Step 1, it sends $(\tilde{z}^t, \tilde{z}^s)$ to $(\tilde{P}^t(\tilde{z}), \tilde{z}^t)$, and $\tilde{P}^t(\tilde{z})$ is strictly decreasing in $\tilde{z}^s$ at every fixed $\tilde{z}^t$ by Lemma I.3 applied to the reparametrized model, whose unobserved components $(\tilde{U}_i^t, \tilde{U}_i^s)$ are an invertible linear transformation of U_i and therefore inherit a continuous, everywhere positive density with the same tail bound. Since the analogous equality holds for the reparametrized model,

$$\begin{aligned} \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^{t,r}, x) &\equiv \mathbb{E}[Y_i \,|\, X_i = x, T_i = t, P_i^t = \tilde{P}^t(\tilde{z}), Z_i^{t,r} = \tilde{z}^{t,r}] \\ &= \mathbb{E}[Y_i \,|\, X_i = x, T_i = t, \tilde{Z}_i = \tilde{z}], \end{aligned}$$

and $\mathbb{E}[Y_i \,|\, X_i = x, T_i = t, \tilde{Z}_i = \tilde{z}] = \mathbb{E}[Y_i \,|\, X_i = x, T_i = t, Z_i = z]$ since $\tilde{Z}_i$ is an invertible linear transformation of Z_i, we can translate the margin-specific regression function to the

[8] An additional requirement for the equality of the two conditional expectations is that the generated sigma-algebras are the same, which is also met because the mapping between z and $(P^t(z), z^{t,r})$ is one-to-one.

reparametrization as follows:

$$m_{ts}(P^t(z), z^{t,r}, x) = \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^{t,r}, x) = \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^t, x),$$

with the second equality following from $\tilde{z}^{t,r} = \tilde{z}^t - \tilde{z}^r = \tilde{z}^t$. Since $(P^t(z), z^{t,r}) = (\tilde{P}^t(\tilde{z}), \tilde{z}^t)$, this shows that $m_{ts}(\cdot)$ and $\tilde{m}_{ts}(\cdot)$ agree as functions on the range of the map $z \mapsto (P^t(z), z^{t,r})$.

Step 3 (Deriving the identity): In the reparametrized model, $\tilde{z}^t$ and $\tilde{z}^s$ enter only their own treatments' assessments, so $(\tilde{Z}_i^t, \tilde{Z}_i^s)$ satisfies unordered partial monotonicity in the sense of Section F.1. The additive structure of the assessments also satisfies Assumption CC of Mountjoy (2022), as noted there. Under these two conditions, Mountjoy (2022, Eq. (14)) shows that the following identity holds:

$$\tilde{\mu}_{ts}(\tilde{z}, x) = \frac{\partial_{\tilde{z}^s} \mathbb{E}[Y_i T_i^t \,|\, X_i = x, \tilde{Z}_i = \tilde{z}]}{\partial_{\tilde{z}^s} \mathbb{E}[T_i^t \,|\, X_i = x, \tilde{Z}_i = \tilde{z}]}.$$

Since the t-treatment propensity depends on x only through $\tilde{z}$, the denominator simplifies to

$$\partial_{\tilde{z}^s} \mathbb{E}[T_i^t \,|\, X_i = x, \tilde{Z}_i = \tilde{z}] = \partial_{\tilde{z}^s} \tilde{P}^t(\tilde{z}) < 0.$$

The strict inequality holds by Lemma I.3 applied to the reparametrized model, whose density lies in the class $\mathcal{F}$ of Section 2.1 as noted in Step 2.

For the numerator, we express the conditional expectation as

$$\begin{aligned}\mathbb{E}[Y_i T_i^t \,|\, X_i = x, \tilde{Z}_i = \tilde{z}] &= \mathbb{E}[Y_i \,|\, X_i = x, T_i^t = 1, \tilde{Z}_i = \tilde{z}] \mathbb{P}(T_i^t = 1 \,|\, X_i = x, \tilde{Z}_i = \tilde{z}) \\ &= \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^t, x) \tilde{P}^t(\tilde{z}).\end{aligned}$$

The derivative in $\tilde{z}^s$ exists by the differentiability hypothesis on m_{ts} and the continuous differentiability of $\tilde{P}^t(\cdot)$, and the product rule gives

$$\begin{aligned}&\partial_{\tilde{z}^s} \mathbb{E}[Y_i T_i^t \,|\, X_i = x, \tilde{Z}_i = \tilde{z}] \\ &= \partial_p \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^t, x) \tilde{P}^t(\tilde{z}) \partial_{\tilde{z}^s} \tilde{P}^t(\tilde{z}) + \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^t, x) \partial_{\tilde{z}^s} \tilde{P}^t(\tilde{z}).\end{aligned}$$

Putting these pieces together and rearranging, we have

$$\tilde{\mu}_{ts}(\tilde{z}, x) = \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^t, x) + \partial_p \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}), \tilde{z}^t, x) \tilde{P}^t(\tilde{z}).$$

Using the established equivalences between the original objects and the reparametrized objects, we can translate this identity back to the original model as follows:

$$\begin{aligned}\mu_{ts}(z,x) &= \tilde{\mu}_{ts}(\tilde{z},x)\\ &= \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}),\tilde{z}^t,x) + \partial_p \tilde{m}_{ts}(\tilde{P}^t(\tilde{z}),\tilde{z}^t,x)\tilde{P}^t(\tilde{z})\\ &= m_{ts}(P^t(z),z^{t,r},x) + P^t(z)\partial_p m_{ts}(P^t(z),z^{t,r},x),\end{aligned}$$

which is the desired identity. □

Proof of Theorem 1. Suppose a candidate pair $\theta = (\boldsymbol{\gamma}, P)$ is observationally equivalent to $\theta_o = (\boldsymbol{\gamma}_o, P_o)$, i.e.,

$$P^t(\gamma_j' x) = P_o^t(\gamma_{oj}' x) \qquad t \in \{0,1,2\},\ x \in \text{supp}(X),\ j \in \{1,\dots,J\}. \tag{A.1}$$

We show (A.1) forces $\boldsymbol{\gamma} = \boldsymbol{\gamma}_o$.

Step 1 (Functional dependence via injectivity): Lemma I.4 applies to $f \in \mathcal{F}$, so $P_o(z) = P(z\,|\,f)$ is an injective function of $z \in \mathbb{R}^2$. Letting $h = P_o^{-1} \circ P$, which depends only on P and P_o and not on j, we have

$$h(\gamma_j' x) = \gamma_{oj}' x \qquad \forall\, x \in \text{supp}(X),\ j = 1,\dots,J. \tag{A.2}$$

The function h above is well-defined at every value $\gamma_j' x$ with $x \in \text{supp}(X)$ and $j = 1,\dots,J$, since the range of $P(\cdot)$ on these values is contained in the range of $P_o(\cdot)$ by (A.1).

Step 2 (Slope alignment on X_a): Fix j and choose $x_b^{j,*} \in \text{supp}(X_b)$ as in Assumption 3(a). Let $B \subseteq \mathbb{R}^{\dim(X_a)}$ be an open ball with $B \subseteq \text{supp}(X_a\,|\,X_b = x_b^{j,*})$. On the slice $\{(X_a, x_b^{j,*}, 1) : X_a \in B\}$, (A.2) reads

$$h(\gamma_{j,d_X} + \gamma_{j,a}' x_a + \gamma_{j,b}' x_b^{j,*}) = \gamma_{oj,d_X} + \gamma_{oj,a}' x_a + \gamma_{oj,b}' x_b^{j,*}, \quad \forall x_a \in B.$$

By the inverse function theorem, $h = P_o^{-1} \circ P$ is continuously differentiable. Differentiating both sides with respect to x_a gives

$$\nabla h(\gamma_{j,d_X} + \gamma_{j,a}' x_a + \gamma_{j,b}' x_b^{j,*}) \cdot \gamma_{j,a}' = \gamma_{oj,a}', \quad \forall x_a \in B.$$

Since $\text{rank}(\gamma_{oj,a}) = 2$ by Assumption 3(b), the above equality implies $\text{rank}(\gamma_{j,a}) = 2$ and

$\operatorname{rank}(\nabla h(\gamma_{j,d_X} + \gamma'_{j,a} x_a + \gamma'_{j,b} x_b^{j,*})) = 2$, and hence

$$\gamma_{oj,a} = \gamma_{j,a} M_j$$

where $M_j = (\gamma'_{j,a}\gamma_{j,a})^{-1}\gamma'_{j,a}\gamma_{oj,a}$ equals $\nabla h(\gamma_{j,d_X} + \gamma'_{j,a} x_a + \gamma'_{j,b} x_b^{j,*})'$ for every $x_a \in B$ and is nonsingular. Letting $\tilde{z} = \gamma_{j,d_X} + \gamma'_{j,a} x_a + \gamma'_{j,b} x_b^{j,*}$, as x_a ranges over B, $\tilde{z}$ ranges over an open subset of $\mathbb{R}^2$ since $\operatorname{rank}(\gamma_{j,a}) = 2$. For any $x_b \in \operatorname{supp}(X_b)$, write $I_j(x_b) := \operatorname{supp}(\gamma'_j X \,|\, X_b = x_b) \subseteq \mathbb{R}^2$, which is the closure of the image of the slice under $x \mapsto \gamma'_j x$. Restricting (A.2) to the slice $X_b = x_b$ and substituting $\gamma_{oj,a} = \gamma_{j,a} M_j$, h takes the affine form $h(\tilde{z}) = M'_j \tilde{z} + c_j^{(x_b)}$ on that image, where

$$c_j^{(x_b)} = (\gamma_{oj,d_X} - M'_j \gamma_{j,d_X}) + (\gamma_{oj,b} - \gamma_{j,b} M_j)' x_b. \tag{A.3}$$

The identity $P(\tilde{z}) = P_o(M'_j \tilde{z} + c_j^{(x_b)})$ extends from the image to its closure by the continuity of P and P_o, and the affine form therefore holds for all $\tilde{z} \in I_j(x_b)$.

Step 3 (Slope alignment on X_b): For $x_b \in \operatorname{supp}(X_b)$, write $I_j^o(x_b) := \operatorname{supp}(\gamma'_{oj} X \,|\, X_b = x_b)$ for the closure of the image of the slice under the true coefficients and $A_{x_b}(\tilde{z}) := M'_j \tilde{z} + c_j^{(x_b)}$ for the affine map of Step 2, so that $h = A_{x_b}$ on $I_j(x_b)$. By (A.2), A_{x_b} maps the image of the slice under $x \mapsto \gamma'_j x$ onto its image under $x \mapsto \gamma'_{oj} x$, and as an affine bijection of $\mathbb{R}^2$ it maps $I_j(x_b)$ onto $I_j^o(x_b)$. Let x_b and x'_b be adjacent slice values. The supports of the observed propensity given the two slices are the closures of $P_o(I_j^o(x_b))$ and $P_o(I_j^o(x'_b))$ by the continuity of P_o, and the two supports share an interior point. By Lemma I.5, that point is interior to $P_o(I_j^o(x_b))$ and to $P_o(I_j^o(x'_b))$, and the same lemma supplies $z_o \in \operatorname{Int} I_j^o(x_b) \cap \operatorname{Int} I_j^o(x'_b)$. Set $\tilde{z} = A_{x_b}^{-1}(z_o)$ and $\tilde{z}' = A_{x'_b}^{-1}(z_o)$. Since A_{x_b} is an affine bijection of $\mathbb{R}^2$, $\tilde{z} \in \operatorname{Int} I_j(x_b)$, and $P(\tilde{z}) = P_o(h(\tilde{z})) = P_o(z_o) = P_o(h(\tilde{z}')) = P(\tilde{z}')$. On $\operatorname{Int} I_j(x_b)$, $P = P_o \circ A_{x_b}$, so that $DP(\tilde{z}) = DP_o(z_o) M'_j$ is nonsingular by Lemma I.3 and the nonsingularity of M_j, hence positive definite by Lemma I.6. By the same lemma, $P(\tilde{z}') = P(\tilde{z})$ forces $\tilde{z}' = \tilde{z}$, that is, $A_{x_b}^{-1}(z_o) = A_{x'_b}^{-1}(z_o)$, which is $c_j^{(x_b)} = c_j^{(x'_b)}$. Iterating this over a chain of adjacent slice values joining $x_b^{j,*}$ to any $x_b \in C_j(x_b^{j,*})$, we have

$$c_j^{(x_b^{j,*})} = c_j^{(x_b)} \qquad \Longleftrightarrow \qquad (\gamma_{oj,b} - \gamma_{j,b} M_j)'(x_b^{j,*} - x_b) = 0$$

for all $x_b \in C_j(x_b^{j,*})$. Invoking Assumption 3(d), we have that $x_b^{j,*} - x_b$ spans $\mathbb{R}^{\dim(X_b)}$ as x_b ranges over $C_j(x_b^{j,*})$, so the above condition implies $\gamma_{oj,b} = \gamma_{j,b} M_j$. Combining with

Step 2, we conclude that

$$\gamma_{oj,-1} = \gamma_{j,-1} M_j, \qquad \gamma_{oj,d_X} = M_j' \gamma_{j,d_X} + c_j, \tag{A.4}$$

where $c_j = \gamma_{oj,d_X} - M_j' \gamma_{j,d_X}$ is the residual examiner-specific intercept after aligning the slopes and, by $\gamma_{oj,b} = \gamma_{j,b} M_j$, equals $c_j^{(x_b)}$ in (A.3) for every $x_b \in \text{supp}(X_b)$. The function h therefore takes the global affine form $h(\tilde{z}) = M_j' \tilde{z} + c_j$ on $\bigcup_{x_b \in \text{supp}(X_b)} I_j(x_b) = \text{supp}(\gamma_j' X) \subseteq \mathbb{R}^2$ and has the range $\text{supp}(\gamma_{oj}' X)$.

Step 4 (M_j is diagonalizable with real eigenvalues): We have shown that $P_o^t(M_j' \tilde{z} + c_j) = P^t(\tilde{z})$ for all $\tilde{z} \in \text{supp}(\gamma_j' X)$ and $t \in \{1, 2\}$. Equivalently, $P_o^t(z) = P^t(M_j^{-T}(z - c_j))$ for all $z \in \text{supp}(\gamma_{oj}' X)$ by substituting $M_j' \tilde{z} + c_j$ with z. Here, $M_j^{-T} = (M_j^{-1})'$ is the inverse transpose of M_j. Differentiating this identity in z and stacking over $t \in \{1, 2\}$, we have

$$DP_o(z) = DP(M_j^{-T}(z - c_j)) M_j^{-T}, \qquad z \in \text{Int}\,(\text{supp}(\gamma_{oj}' X)),$$

where $DP_o(\cdot)$ and $DP(\cdot)$ are Jacobians of $P_o(\cdot)$ and $P(\cdot)$, respectively.

Taking the transpose of $DP(M_j^{-T}(z - c_j)) = DP_o(z) M_j'$ and using the symmetry of $DP(\cdot)$ and $DP_o(\cdot)$, we obtain $DP(M_j^{-T}(z - c_j)) = M_j DP_o(z)$. Subtracting the two identities, we obtain

$$DP_o(z) M_j' = M_j DP_o(z).$$

Fix any $z_0 \in \text{Int}\,(\text{supp}(\gamma_{oj}' X))$ and let $H = DP_o(z_0)$. The displayed identity at z_0 reads $M_j H = H M_j'$, with H symmetric positive definite. We show that M_j is diagonalizable with real eigenvalues. Multiplying $M_j H = H M_j'$ on the left by $H^{-1/2}$ and on the right by $H^{-1/2}$ gives,

$$H^{-1/2} M_j H^{1/2} = H^{1/2} M_j' H^{-1/2}.$$

The right-hand side is the transpose of the left-hand side, so $H^{1/2} M_j' H^{-1/2}$ equals its own transpose, i.e., is symmetric. A symmetric real matrix is diagonalizable with real eigenvalues. Since $M_j = H^{1/2}(H^{-1/2} M_j H^{1/2}) H^{-1/2}$ is similar to the symmetric matrix $H^{-1/2} M_j H^{1/2}$, M_j inherits diagonalizability and real spectrum. Write

$$M_j = U_j D_j U_j^{-1},$$

with U_j and $D_j = \text{diag}(d_{j,1}, d_{j,2}) \in \mathbb{R}^{2 \times 2}$ both invertible.

Step 5 (Separability obstruction): Substituting $M_j = U_j D_j U_j^{-1}$ into the condition

$M_j DP_o(z) = DP_o(z)M_j'$ from Step 4 gives, after multiplying on the left by U_j^{-1} and on the right by $U_j^{-T} = (U_j')^{-1}$,

$$D_j U_j^{-1} DP_o(z) U_j^{-T} = U_j^{-1} DP_o(z) U_j^{-T} D_j \qquad \textit{for all } z \in \operatorname{Int}(\operatorname{supp}(\gamma_{oj}' X)).$$

Suppose, for contradiction, that $d_{j,1} \neq d_{j,2}$. Then any symmetric matrix commuting with D_j must be diagonal. In particular, $U_j^{-1} DP_o(z) U_j^{-T}$ is diagonal at every $z \in \operatorname{Int}(\operatorname{supp}(\gamma_{oj}' X))$. Now let v_1, v_2 denote the columns of U_j^{-T}, and reparametrize $z = U_j^{-T}\check{z}$, so that $\check{z} = U_j' z$. Define $\tilde{G}_o(\check{z}) = G_o(U_j^{-T}\check{z})$ where $G_o(\cdot) = G(\cdot \,|\, f)$. The Hessian transforms as

$$\nabla^2_{\check{z}} \tilde{G}_o(\check{z}) = U_j^{-1} DP_o(U_j^{-T}\check{z}) U_j^{-T},$$

which is diagonal on the corresponding open subset by the previous display. Pulling back to z,

$$0 \equiv \frac{\partial^2}{\partial \check{z}^1 \partial \check{z}^2} \tilde{G}_o(\check{z}) = v_1' \nabla^2 G_o(U_j^{-T}\check{z}) v_2 = v_1' DP_o(z) v_2, \qquad z \in \operatorname{Int}(\operatorname{supp}(\gamma_{oj}' X)).$$

This is exactly the separability ruled out by Assumption 3(c)—a contradiction. Hence $d_{j,1} = d_{j,2} =: d_j$ and $M_j = d_j \mathbf{I}_2$.

It remains to fix the sign of $d_j \neq 0$. With $M_j = d_j \mathbf{I}_2$, the level identity of Step 4 reads $P_o(d_j \tilde{z} + c_j) = P(\tilde{z})$, and differentiating in $\tilde{z}$ gives

$$DP(\tilde{z}) = d_j DP_o(d_j \tilde{z} + c_j), \qquad \tilde{z} \in \operatorname{Int}(\operatorname{supp}(\gamma_j' X)).$$

Here $DP_o(\cdot)$ is positive definite by Lemma I.3, and $DP(\tilde{z})$ is positive semidefinite by Lemma I.6 since $(\boldsymbol{\gamma}, P)$ is a candidate pair. If $d_j < 0$, the right-hand side would be negative definite, contradicting the positive semidefiniteness of the left-hand side. Since M_j is nonsingular we have $d_j \neq 0$, and therefore $d_j > 0$. Consequently $DP(\cdot) = d_j DP_o(d_j \cdot + c_j)$ is positive definite on $\operatorname{Int}(\operatorname{supp}(\gamma_j' X))$ for every j, so P is injective on $\cup_{j=1}^J \operatorname{Int}(\operatorname{supp}(\gamma_j' X))$ by Lemma I.6. Returning to the within-examiner identities,

$$\gamma_{oj,-1} = d_j \gamma_{j,-1}, \qquad \gamma_{oj,d_X} = d_j \gamma_{j,d_X} + c_j,$$

for some examiner-specific scalar $d_j > 0$ and intercept $c_j \in \mathbb{R}^2$.

Step 6 (Between-examiner comparison and location determination): Take any pair

of adjacent examiners j_1, j_2, and set $\mathcal{O} := \mathrm{Int}\,(\mathrm{supp}(\gamma'_{o,j_1}X)) \cap \mathrm{Int}\,(\mathrm{supp}(\gamma'_{o,j_2}X))$. By adjacency, the propensity supports share an interior point. These supports are the closures of $P_o(\mathrm{supp}(\gamma'_{o,j_1}X))$ and $P_o(\mathrm{supp}(\gamma'_{o,j_2}X))$ by the continuity of P_o, and Lemma I.5 places that shared point in $P_o(\mathcal{O})$. Hence $\mathcal{O}$ is nonempty, and as an intersection of open sets it is open. Fix $z_0 \in \mathcal{O}$. By Step 5 applied to each examiner, the level identity at z_0 reads

$$P^t(d_{j_1}^{-1}(z_0 - c_{j_1})) = P_o^t(z_0) = P^t(d_{j_2}^{-1}(z_0 - c_{j_2})) \qquad \forall\, t \in \{1, 2\}.$$

The affine map $\xi \mapsto d_{j_i}^{-1}(\xi - c_{j_i})$ carries $\mathrm{Int}\,(\mathrm{supp}(\gamma'_{o,j_i}X))$ onto $\mathrm{Int}\,(\mathrm{supp}(\gamma'_{j_i}X))$, so the two arguments above lie in $\cup_{j=1}^{J}\mathrm{Int}\,(\mathrm{supp}(\gamma'_j X))$, where DP is positive definite. By Lemma I.6, $P(\cdot)$ is injective there, so

$$d_{j_1}^{-1}(z_0 - c_{j_1}) = d_{j_2}^{-1}(z_0 - c_{j_2}).$$

Rearranging,

$$(d_{j_1}^{-1} - d_{j_2}^{-1})z_0 = d_{j_1}^{-1}c_{j_1} - d_{j_2}^{-1}c_{j_2} \in \mathbb{R}^2.$$

The same equation evaluated at any other point $z'_0 \in \mathcal{O}$ gives $(d_{j_1}^{-1} - d_{j_2}^{-1})z'_0 = d_{j_1}^{-1}c_{j_1} - d_{j_2}^{-1}c_{j_2}$. Subtracting,

$$(d_{j_1}^{-1} - d_{j_2}^{-1})(z_0 - z'_0) = 0.$$

Since $\mathcal{O}$ is open and nonempty, it contains two distinct points, so this forces $d_{j_1} = d_{j_2}$. By the connectedness of the examiner-adjacency graph, iterating this argument over the path joining j_1 to j_2 in the examiner-adjacency graph, we have $d_{j_1} = d_{j_2}$ for every pair of connected examiners, and hence $d_j \equiv d$ for some scalar d across all examiners. Substituting this back into the level identity at z_0, we have $c_{j_1} = c_{j_2}$ for every pair of connected examiners by the same argument, so $c_j \equiv c$ for a common $c \in \mathbb{R}^2$. Summing the intercept identities $\gamma_{oj,d_X} = d\gamma_{j,d_X} + c$ over examiners gives $\sum_j \gamma_{oj,d_X} = d\sum_j \gamma_{j,d_X} + Jc$. Both sums vanish by the location normalization $\boldsymbol{e}'\boldsymbol{\gamma}^t = \boldsymbol{e}'\boldsymbol{\gamma}_o^t = 0$, whence $c = 0$.

Step 7 (Scale determination): From Step 6, $\gamma_{oj} = d\gamma_j$ for every j. The constraint $|||\boldsymbol{\gamma}||| = 1 = |||\boldsymbol{\gamma}_o|||$ gives

$$1 = |||\boldsymbol{\gamma}_o|||^2 = \sum_{j=1}^{J} |||\gamma_{oj}|||^2 = \sum_{j=1}^{J} |||d\gamma_j|||^2 = d^2 \sum_{j=1}^{J} |||\gamma_j|||^2 = d^2|||\boldsymbol{\gamma}|||^2 = d^2,$$

and hence $|d| = 1$, i.e., $d = \pm 1$. Using $d > 0$ from Steps 5 and 6, we have $d = 1$, so $\gamma_{oj} = \gamma_j$ for every j. This establishes $\boldsymbol{\gamma}_o = \boldsymbol{\gamma}$, i.e., the first-stage coefficient matrix is identified.

Step 8 (Identification of P_o): With $\boldsymbol{\gamma} = \boldsymbol{\gamma}_o$, (A.1) and the continuity of P and P_o give

$$P^t(z) = P_o^t(z), \qquad z \in \text{supp}(\boldsymbol{\gamma}_o' W),\ t \in \{0, 1, 2\}.$$

Thus $P_o(\cdot)$ is identified as a function of $z \in \text{supp}(\boldsymbol{\gamma}_o' W)$. This completes the proof. □

B Consistency and the Crude Rate of Convergence

This section proves Theorems 2(i) and 2(ii). The argument runs as follows. Lemmas B.1–B.5 collect the preliminary inputs: the compact closure of $\mathcal{P}$, an entropy bound, a stability bound, the simultaneous spline approximation of (P_o, G_o), and metric comparisons. Proposition B.1 derives the nonparametric rate $\mathsf{H}(\hat{\theta}, \theta_o) = O_{\mathrm{P}}(\rho_N)$ from these inputs via the rate theorem of Van der Vaart and Wellner (1996, Theorem 3.4.1). Lemma B.6 converts Theorem 1 into a global separation statement in the L metric, which combined with the rate proves Theorem 2(i). Lemmas B.7–B.10 and Propositions B.2 and B.3 then establish the local curvature bound $\mathsf{L} \gtrsim |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o|||$, which proves the first claim of Theorem 2(ii). Proposition B.4 addresses the uniform rate of $P_{\hat{\xi}}$.

Conventions. Throughout this section, $\Theta_N := \Gamma \times \{P_\xi : \xi \in \Xi(K_N)\}$ denotes the feasible set of the SMLE problem (13), and $\theta_K := (\boldsymbol{\gamma}_o, P_{\xi_K^\circ})$ with ξ_K° the approximant of Lemma B.4. We write $d_K = (K + q - 1)^2$ and $R_W := \sup_{w \in \text{supp}(W_i)} \|w\|$, which is finite because X_i has compact support. Since $\bar{\mathcal{Z}} \supseteq \cup_{\boldsymbol{\gamma} \in \Gamma} \text{supp}(\boldsymbol{\gamma}' W_i)$, every realized index $\boldsymbol{\gamma}' W_i$ with $\boldsymbol{\gamma} \in \Gamma$ lies in $\bar{\mathcal{Z}}$, so every criterion value below is well defined. Under Assumption 4(c), $q \geq 4$, a sieve propensity P_ξ with $\xi \in \Xi(K)$ is a derivative of an order-(q, q) tensor spline with simple knots and is therefore $C^{q-3} \subseteq C^1$. Since the constraints in $\Xi(K)$ reproduce the remaining defining conditions of $\mathcal{P}$, it follows that $\text{GradSpl}(K) = \{P_\xi : \xi \in \Xi(K)\} \subseteq \mathcal{P}$. Throughout, the real $k_o \geq 2$ denotes the smoothness index of P_o (Assumption 4(b)), so that $\rho_N = \sqrt{K_N^2 \log N / N + K_N^{-k_o}}$. The approximation theory (Lemma B.4) uses $q \geq k_o + 1$, also granted by Assumption 4(c).

Lemma B.1 (closure of the propensity class)**.** *Let $\bar{\mathcal{P}}$ denote the closure of $\mathcal{P}$ in the sup norm on the continuous maps $\bar{\mathcal{Z}} \to \mathbb{R}^2$. Then $\bar{\mathcal{P}}$ is compact, and every $P \in \bar{\mathcal{P}}$ satisfies the cap, is $2M$-Lipschitz, is the gradient of a C^1 surplus function on $\bar{\mathcal{Z}}$, and is monotone in the sense of $(z - z')'(P(z) - P(z')) \geq 0$.*

Proof. For $P \in \mathcal{P}$: the cap holds by definition; the diagonal entries of DP lie in $[0, M]$ and dominate the common off-diagonal entry in absolute value, so $\|DP\| \leq 2M$ pointwise and

P is $2M$-Lipschitz on the convex $\bar{\mathcal{Z}}$; DP is symmetric, so line integrals of P do not depend on the path and P is the gradient of a C^1 surplus function on the rectangle $\bar{\mathcal{Z}}$; and DP is positive semidefinite by Lemma I.6, so the line-integral identity in the proof of Lemma I.6 gives monotonicity. Each of the four properties passes to uniform limits (line integrals along fixed paths and the monotonicity inequality converge directly, so path independence and the surplus representation are preserved), so they hold on $\bar{\mathcal{P}}$. Compactness follows from the Arzelà–Ascoli theorem: $\bar{\mathcal{P}}$ is uniformly bounded, uniformly $2M$-Lipschitz, and closed. □

Lemma B.2 (entropy of the sieve criterion class)**.** *For $t \in \{1,2\}$ let $\mathcal{G}_K^t = \{z \mapsto \partial_{z^t} b_K(z)'\xi : \xi \in \Xi(K)\}$, and let $\mathcal{L}_K := \{(t,w) \mapsto \sum_t t^t \log P_\xi^t(\boldsymbol{\gamma}' w) : \boldsymbol{\gamma} \in \Gamma, \xi \in \Xi(K)\}$ be the criterion class. Then, for $0 < \epsilon \leq 1$:*

(i) $\sup_Q \log N(\epsilon, \mathcal{G}_K^t, L^2(Q)) \lesssim d_K \log(1/\epsilon)$*, the supremum over finitely supported probability measures.*

(ii) $\mathcal{L}_K$ *has the constant envelope* $\log(1/\kappa)$ *and* $\sup_Q \log N(\epsilon, \mathcal{L}_K, L^2(Q)) \lesssim d_K \log(C/\epsilon)$ *for a constant* $C = C(\kappa, M, R)$*.*

(iii) $\mathbb{E} \sup_{\theta \in \Theta_N} |(\mathbb{P}_N - \mathbb{P}) l_i(\theta)| \lesssim \sqrt{d_{K_N}/N}$*.*

Proof. (i) $\mathcal{G}_K^t$ lies in the d_K-dimensional vector space $\{\partial_{z^t} b_K(\cdot)'h : h \in \mathbb{R}^{d_K}\}$ and has envelope 1 by the cap. A finite-dimensional vector space of functions is VC-subgraph of index at most $d_K + 2$ (Van der Vaart and Wellner, 1996, Lemma 2.6.15), and the uniform entropy bound for VC-subgraph classes with bounded envelope (Van der Vaart and Wellner, 1996, Theorem 2.6.7) gives the claim. Restricting to the convex set $\Xi(K)$ only shrinks the class.

(ii) On $[\kappa, 1-\kappa]$ the logarithm is κ^{-1}-Lipschitz. Fix $\boldsymbol{\gamma}$ and ξ, ξ'. Then $|l(t,w;\boldsymbol{\gamma},\xi) - l(t,w;\boldsymbol{\gamma},\xi')| \leq \kappa^{-1} \sum_{t \in \{1,2,0\}} |P_\xi^t(\boldsymbol{\gamma}' w) - P_{\xi'}^t(\boldsymbol{\gamma}' w)|$, and the 0-component is the sum of the other two, so covering each $\mathcal{G}_K^t$ to accuracy $\epsilon\kappa/6$ under the pushforward of Q by $w \mapsto \boldsymbol{\gamma}' w$ covers the ξ-direction. By (i) this costs $\lesssim d_K \log(1/\epsilon)$ uniformly in $\boldsymbol{\gamma}$ and Q. Fix ξ and $\boldsymbol{\gamma}, \boldsymbol{\gamma}'$. Then by Lemma B.3 and boundedness of W, $|l(t,w;\boldsymbol{\gamma},\xi) - l(t,w;\boldsymbol{\gamma}',\xi)| \leq \kappa^{-1} \cdot 3 \cdot 2MR_W |||\boldsymbol{\gamma} - \boldsymbol{\gamma}'|||$. Since Γ is a compact subset of a fixed-dimensional Euclidean space, an $\epsilon\kappa/(12MR_W)$-net of Γ has log-cardinality $\lesssim \log(C/\epsilon)$. Combining the two nets gives (ii). The envelope is $|\log \kappa|$ since $P^t \geq \kappa$.

(iii) The uniform entropy integral of $\mathcal{L}_{K_N}$ satisfies

$$\tilde{J}(1, \mathcal{L}_{K_N}) \lesssim \int_0^1 \sqrt{1 + d_{K_N} \log(C/\epsilon)} d\epsilon \lesssim \sqrt{d_{K_N}},$$

so the maximal inequality $\mathbb{E}\|\mathbb{G}_N\|_{\mathcal{L}_{K_N}} \lesssim \tilde{J}(1, \mathcal{L}_{K_N})\|F\|_{L^2(\mathbb{P})}$ with the constant envelope $F = |\log \kappa|$ gives $\mathbb{E} \sup_{\Theta_N} |(\mathbb{P}_N - \mathbb{P}) l_i| \lesssim \sqrt{d_{K_N}/N}$. □

Lemma B.3 (stability of the sieve propensities)**.** *For every K, every $\xi \in \Xi(K)$, and $t \in \{1, 2\}$:*

(i) $\sup_z \|DP_\xi^t(z)\| \le 2M$ *and* P_ξ^t *is* $2M$*-Lipschitz on* $\bar{\mathcal{Z}}$*.*

(ii) If $q \ge 4$, as Assumption 4(c) implies, then $\sup_z \|D^2 P_\xi^t(z)\| \le C(q, c_1, c_2)MK$*.*

The same bounds in (i) hold for every $P \in \mathcal{P}$.

Proof. (i) The constraints in $\Xi(K)$ bound the diagonal entries of DP_ξ by M and, through diagonal dominance, the off-diagonal entry by the diagonal ones, so every entry of DP_ξ^t is bounded by M in absolute value. The Lipschitz bound follows by integrating along segments of the convex $\bar{\mathcal{Z}}$. The same argument applies to $P \in \mathcal{P}$.

(ii) Each entry of DP_ξ^t is a tensor spline of coordinate orders at least $q - 2 \ge 2$ on the quasi-uniform mesh, so the Markov inequality for splines (DeVore and Lorentz, 1993, Ch. 5, Theorem 1.2) applied at $\ell = 1$, $p = \infty$ gives $\|D(DP_\xi^t)\|_{L^\infty} \le C(q)\delta_0^{-1}\|DP_\xi^t\|_{L^\infty}$ with $\delta_0 \asymp K^{-1}$ by Assumption 4(c). Combined with (i) this yields the claim. □

Lemma B.4 (simultaneous approximation and feasibility)**.** *Let $G_o = G(\cdot \mid f)$, so that $P_o = \nabla G_o$ and $G_o \in C^{k_o+1}(\bar{\mathcal{Z}})$, and let $q \ge k_o + 1$. There exists $\xi^\circ = \xi_K^\circ \in \mathbb{R}^{d_K}$ such that*

$$\|D^i(G_o - b_K'\xi^\circ)\|_{L^\infty(\bar{\mathcal{Z}})} \lesssim K^{-(k_o+1-i)}, \quad 0 \le i \le \lfloor k_o \rfloor,$$
$$\|D^j(P_o^t - P_{\xi^\circ}^t)\|_{L^\infty(\bar{\mathcal{Z}})} \lesssim K^{-(k_o-j)}, \quad 0 \le j \le \lfloor k_o \rfloor - 1,$$

for $t \in \{1, 2\}$. Moreover, $\xi^\circ \in \Xi(K)$ for all K large enough.

Proof. Approximation. Set $b_K'\xi^\circ := Q_K G_o$, where $Q_K = Q^{(1)}Q^{(2)}$ is the tensor product of the univariate de Boor–Fix quasi-interpolant Q of order q, acting coordinatewise on the box $\bar{\mathcal{Z}}$. On a quasi-uniform mesh Q is local, reproduces polynomials of degree $\le q - 1$, and approximates a function together with its derivatives: for $g \in C^m(I)$ with $0 \le a \le m \le q - 1$,

$$\|D^a(g - Qg)\|_{L^\infty(I)} \lesssim h^{m-a}\, \omega(D^m g, h), \tag{B.1}$$

with $h \asymp K^{-1}$ and $\omega(f,\cdot)$ the modulus of continuity of f, and for $g \in C^q(I)$ and $0 \le a \le q-1$ it holds at $m=q$ with $\|D^q g\|_{L^\infty(I)}$ in place of $\omega(D^q g, h)$ (De Boor, 1978, Ch. XII). See also Schumaker (2007, Theorem 6.27). When $D^m g$ is Hölder continuous with exponent β, $\omega(D^m g, h) \lesssim h^\beta$ and the bound reads $h^{m+\beta-a}$. Since $\bar{\mathcal{Z}}$ is a product and each $Q^{(\nu)}$ acts on one coordinate, the tensor quasi-interpolant inherits (B.1) on $\bar{\mathcal{Z}}$ (De Boor, 1978, Ch. XVII). With $G_o \in C^{k_o+1}(\bar{\mathcal{Z}})$ and $q \ge k_o+1$, applied at $m = \lfloor k_o \rfloor + 1$ and $\beta = k_o - \lfloor k_o \rfloor$, which satisfy $m \le q-1$ when $\beta > 0$ and $m \le q$ when $\beta = 0$, this gives the first display. The second follows since $D^j(P_o^t - P_{\xi^\circ}^t) = D^j \partial_{z^t}(G_o - b_K' \xi^\circ)$ is the surplus bound at $i = j+1$.

Feasibility. Write $\epsilon_K := \max_t \|P_o^t - P_{\xi^\circ}^t\|_{L^\infty(\bar{\mathcal{Z}})} + \max_t \|D(P_o^t - P_{\xi^\circ}^t)\|_{L^\infty(\bar{\mathcal{Z}})} \lesssim K^{-(k_o-1)} \to 0$. Each constraint in $\Xi(K)$ holds for P_o with a uniform margin on the compact $\bar{\mathcal{Z}}$: the cap with margin by Assumption 4(b) and continuity; $\|DP_o\| < M$ with margin likewise; the diagonal entries of DP_o and the dominance gaps are continuous and strictly positive by Lemma I.3, hence bounded away from zero; and $\partial_{z^2} P_o^1(z) = -\int_{-\infty}^{z^1} f(u^1, z^2 - z^1 + u^1) du^1$ is continuous and strictly negative by Lemma I.2 and the positivity of densities in $\mathcal{F}$, hence bounded away from zero. Since every constraint is a level or first-derivative functional of the propensity, P_{ξ° violates none of them once ϵ_K is smaller than the smallest margin, so $\xi^\circ \in \Xi(K)$ for K large. □

Lemma B.5 (reverse metric comparison under the cap)**.** *Let $\theta = (\boldsymbol{\gamma}, P)$ and $\tilde{\theta} = (\tilde{\boldsymbol{\gamma}}, \tilde{P})$ have propensities bounded below by κ along the relevant indices. Then $\mathrm{KL}(\theta \| \tilde{\theta}) \le \kappa^{-1} \mathsf{L}^2(\theta, \tilde{\theta})$. In particular, $\mathrm{KL}(\theta_o \| \theta_K) \le 3\kappa^{-1} \max_t \|P_o^t - P_{\xi^\circ}^t\|_{L^\infty(\bar{\mathcal{Z}})}^2 \lesssim K^{-2k_o}$ and $\mathsf{H}(\theta_o, \theta_K) \lesssim K^{-k_o}$.*

Proof. Pointwise in w, with $p^t := P^t(\boldsymbol{\gamma}' w)$ and $q^t := \tilde{P}^t(\tilde{\boldsymbol{\gamma}}' w)$, the inequality $\log(1+t) \le t$ gives $\sum_t p^t \log(p^t/q^t) \le \sum_t p^t (p^t - q^t)/q^t = \sum_t (p^t - q^t)^2/q^t \le \kappa^{-1} \sum_t (p^t - q^t)^2$, where the middle equality uses $\sum_t (p^t - q^t) = 0$. Taking expectations proves the first claim. The second follows since θ_K shares the index $\boldsymbol{\gamma}_o$, so $\mathsf{L}^2(\theta_o, \theta_K) \le 3 \max_t \|P_o^t - P_{\xi^\circ}^t\|_\infty^2$, and $\mathsf{H}^2 \le \mathrm{KL}$ by Lemma I.1. □

Proposition B.1 (nonparametric rate)**.** *Suppose Assumptions 4(a)–(c) hold, $K_N \to \infty$ with $K_N^2 \log N = o(N)$, and $\xi_{K_N}^\circ \in \Xi(K_N)$ for all large N (Lemma B.4). Then $\mathsf{H}(\hat{\theta}, \theta_o) = O_{\mathrm{P}}(\rho_N)$ and $\mathsf{L}(\hat{\theta}, \theta_o) = O_{\mathrm{P}}(\rho_N)$.*

Proof. We verify the conditions of Van der Vaart and Wellner (1996, Theorem 3.4.1) for the criterion $\mathbb{M}_N(\theta) := \mathcal{L}_N(\theta) = \mathbb{P}_N l_i(\theta)$, its population version $\mathbb{M}(\theta) := \mathbb{E} l_i(\theta)$, the semimetric $d = \mathsf{H}$ centered at θ_o, and the feasible set Θ_N.

Step 0: approximate maximization and consistency. Since $\hat{\theta}$ maximizes $\mathcal{L}_N$ over $\Theta_N \ni \theta_{K_N}$,

$$\mathbb{M}_N(\hat{\theta}) \geq \mathbb{M}_N(\theta_{K_N}) = \mathbb{M}_N(\theta_o) - \mathbb{P}_N\left(l_i(\theta_o) - l_i(\theta_{K_N})\right). \tag{B.2}$$

The term $h_N := l_i(\theta_o) - l_i(\theta_{K_N})$ is a fixed function for each N with mean $\mathrm{KL}(\theta_o \| \theta_{K_N})$ and, by the κ^{-1}-Lipschitz property of the logarithm and Lemma I.1, variance $\lesssim \mathsf{H}^2(\theta_o, \theta_{K_N})$. Chebyshev's inequality and Lemma B.5 give

$$\mathbb{P}_N h_N = \mathrm{KL}(\theta_o \| \theta_{K_N}) + O_{\mathrm{P}}\left(N^{-1/2}\mathsf{H}(\theta_o, \theta_{K_N})\right) = O(K_N^{-2k_o}) + O_{\mathrm{P}}(N^{-1/2}K_N^{-k_o}) = O_{\mathrm{P}}(\rho_N^2),$$

using $N^{-1/2}K_N^{-k_o} \leq \frac{1}{2}(N^{-1} + K_N^{-2k_o}) \leq \rho_N^2$. For consistency, $\mathbb{M}(\theta) - \mathbb{M}(\theta_o) = -\mathrm{KL}(\theta_o \| \theta) \leq -\mathsf{H}^2(\theta, \theta_o)$ for every $\theta \in \Theta_N$ by Lemma I.1, so

$$\begin{aligned}\mathsf{H}^2(\hat{\theta}, \theta_o) &\leq \mathbb{M}(\theta_o) - \mathbb{M}(\hat{\theta}) \leq \mathbb{M}_N(\theta_o) - \mathbb{M}_N(\hat{\theta}) + 2\sup_{\Theta_N} |(\mathbb{P}_N - \mathbb{P})l_i| \\ &\leq \mathbb{P}_N h_N + 2\sup_{\Theta_N} |(\mathbb{P}_N - \mathbb{P})l_i|,\end{aligned}$$

and the right side is $o_{\mathrm{P}}(1)$ by Lemma B.2(iii) and $d_{K_N} = o(N)$. Hence $\mathsf{H}(\hat{\theta}, \theta_o) = o_{\mathrm{P}}(1)$.

Step 1: curvature. As displayed above, $\mathbb{M}(\theta) - \mathbb{M}(\theta_o) \leq -\mathsf{H}^2(\theta, \theta_o)$ on Θ_N.

Step 2: modulus of continuity. Let $\mathcal{M}_\delta := \{l_i(\theta) - l_i(\theta_o) : \theta \in \Theta_N, \mathsf{H}(\theta, \theta_o) \leq \delta\}$. By the Lipschitz property of the logarithm and Lemma I.1, $\mathrm{Var}(l_i(\theta) - l_i(\theta_o)) \leq 3\kappa^{-2}\mathsf{L}^2(\theta, \theta_o) \leq 12\kappa^{-2}\mathsf{H}^2(\theta, \theta_o)$, so the class $\mathcal{M}_\delta$ has second moments bounded by $\sigma^2 \asymp \delta^2$, the constant envelope $2|\log \kappa|$, and the uniform entropy of Lemma B.2(ii). The maximal inequality of Chernozhukov et al. (2014, Theorem 5.2) yields

$$\mathbb{E}\sup_{\mathcal{M}_\delta} |\mathbb{G}_N(l_i(\theta) - l_i(\theta_o))| \lesssim \phi_N(\delta) := \delta\sqrt{d_{K_N}\log(A/\delta)} + \frac{d_{K_N}\log(A/\delta)}{\sqrt{N}}$$

for a constant A, and $\delta \mapsto \phi_N(\delta)/\delta^{3/2}$ is decreasing on $(0, A/e)$.

Step 3: rate. Since $\log(A/\rho_N) \lesssim \log N$, we have

$$\rho_N^{-2}\phi_N(\rho_N) \lesssim \rho_N^{-1}\sqrt{d_{K_N}\log N} + \rho_N^{-2}\frac{d_{K_N}\log N}{\sqrt{N}} \lesssim \sqrt{N},$$

where both bounds use $\rho_N^2 \geq K_N^2 \log N/N \gtrsim d_{K_N}\log N/N$. By Step 0, $\mathbb{M}_N(\hat{\theta}) \geq \mathbb{M}_N(\theta_o) - O_{\mathrm{P}}(\rho_N^2)$ and $\mathsf{H}(\hat{\theta}, \theta_o) = o_{\mathrm{P}}(1)$. Theorem 3.4.1 of Van der Vaart and Wellner (1996) then gives $\mathsf{H}(\hat{\theta}, \theta_o) = O_{\mathrm{P}}(\rho_N)$, and $\mathsf{L} \leq 2\mathsf{H}$ (Lemma I.1) completes the proof. □

Lemma B.6 (global identification in the L metric)**.** *Suppose Assumptions 1, 3 and 4(a) hold. Then, for every $\epsilon > 0$,*

$$\delta(\epsilon) := \inf \left\{ \mathsf{L}\left((\boldsymbol{\gamma}, P), \theta_o\right) : \boldsymbol{\gamma} \in \Gamma, \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_o\| \geq \epsilon, P \in \bar{\mathcal{P}} \right\} > 0.$$

Proof. Suppose not. Then there are $(\boldsymbol{\gamma}_n, P_n) \in \Gamma \times \bar{\mathcal{P}}$ with $\|\boldsymbol{\gamma}_n - \boldsymbol{\gamma}_o\| \geq \epsilon$ and $\mathsf{L}((\boldsymbol{\gamma}_n, P_n), \theta_o) \to 0$. By compactness of Γ and of $\bar{\mathcal{P}}$ (Lemma B.1), pass to a subsequence with $\boldsymbol{\gamma}_n \to \boldsymbol{\gamma}^* \in \Gamma$, $\|\boldsymbol{\gamma}^* - \boldsymbol{\gamma}_o\| \geq \epsilon$, and $P_n \to \bar{P} \in \bar{\mathcal{P}}$ uniformly. Since $|P_n^t(\boldsymbol{\gamma}_n' w) - \bar{P}^t(\boldsymbol{\gamma}^{*\prime} w)| \leq \|P_n - \bar{P}\|_\infty + 2MR_W\|\boldsymbol{\gamma}_n - \boldsymbol{\gamma}^*\| \to 0$ uniformly over $\mathrm{supp}(W)$, we get $\mathsf{L}((\boldsymbol{\gamma}^*, \bar{P}), \theta_o) = 0$, i.e.

$$\bar{P}^t(\boldsymbol{\gamma}^{*\prime} w) = P_o^t(\boldsymbol{\gamma}_o' w) \quad \textit{for } \mathbb{P}\textit{-almost every } w, t \in \{0, 1, 2\}. \tag{B.3}$$

Both sides of (B.3) are continuous in w, and every point of $\mathrm{supp}(W)$ is a limit of points where (B.3) holds (the equality set has full measure, hence is dense in the support), so (B.3) holds for *every* $w \in \mathrm{supp}(W)$. By Assumption 4(a), $\lambda_j > 0$ for every j, so this is the observational equivalence (A.1) between $(\boldsymbol{\gamma}^*, \bar{P})$ and $(\boldsymbol{\gamma}_o, P_o)$.

It remains to show that observational equivalence forces $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}_o$, contradicting $\|\boldsymbol{\gamma}^* - \boldsymbol{\gamma}_o\| \geq \epsilon$. Theorem 1 does not apply verbatim, because $\bar{P} \in \bar{\mathcal{P}}$ is only Lipschitz. We rerun the proof of Theorem 1, replacing its differentiation step by a linear-algebra argument and recovering smoothness from the resulting affine form. The remaining steps carry over.

Linear algebra in place of differentiation. $h := P_o^{-1} \circ \bar{P}$ is well defined and continuous on $\cup_j \mathrm{supp}(\gamma_j^{*\prime} X)$, and $h(\gamma_j^{*\prime} x) = \gamma_{oj}' x$ on $\mathrm{supp}(X)$ for every j, where γ_j^* denotes judge j's block of $\boldsymbol{\gamma}^*$. Fix j and the slice ball $B \subseteq \mathrm{supp}(X_a \,|\, X_b = x_b^{j,*})$ of Assumption 3(a). On the slice, $h(\gamma_{j,1}^* + \gamma_{j,a}^{*\prime} x_a + \gamma_{j,b}^{*\prime} x_b^{j,*}) = \gamma_{oj,1} + \gamma_{oj,a}' x_a + \gamma_{oj,b}' x_b^{j,*}$ for all $x_a \in B$. If $\gamma_{j,a}^{*\prime}(x_a - \tilde{x}_a) = 0$ for $x_a, \tilde{x}_a \in B$, the left sides coincide, hence so do the right sides: $\ker(\gamma_{j,a}^{*\prime}) \cap (B - B) \subseteq \ker(\gamma_{oj,a}')$. Since $B - B$ contains an open neighborhood of the origin and $\mathrm{rank}(\gamma_{oj,a}) = 2$ by Assumption 3(b), this forces $\mathrm{rank}(\gamma_{j,a}^*) = 2$. Choose a linear right inverse S with $\gamma_{j,a}^{*\prime} S = \mathbf{I}_2$. For $\tilde{z}$ in the open image $I_j(x_b^{j,*})$ of the slice, write $x_a = S(\tilde{z} - \gamma_{j,1}^* - \gamma_{j,b}^{*\prime} x_b^{j,*}) + \nu$ with $\nu \in \ker(\gamma_{j,a}^{*\prime})$. The kernel inclusion shows $\gamma_{oj,a}' \nu = 0$, so

$$h(\tilde{z}) = M_j' \tilde{z} + c_j^{(x_b^{j,*})}, \qquad M_j' := \gamma_{oj,a}' S,$$

on $I_j(x_b^{j,*})$, with $c_j^{(x_b^{j,*})}$ as in (A.3) and $\gamma_{oj,a} = \gamma_{j,a}^* M_j$, M_j nonsingular.

The slice-pooling argument of Step 3 uses the level identity, Lemma I.5, and the injectivity of $\bar{P}$ at an interior point of a slice image, where $\bar{P} = P_o \circ A_{x_b}$ is C^1 with

nonsingular Jacobian and the separation argument of Lemma I.6 applies to the monotone map $\bar{P}$. It therefore goes through and yields $\gamma_{oj,-1} = \gamma^*_{j,-1} M_j$ and the global affine form $h(\tilde{z}) = M_j' \tilde{z} + c_j$ on $\mathrm{supp}(\gamma_j^{*\prime} X)$.

Smoothness from the affine form. The level identity now reads $\bar{P}(\tilde{z}) = P_o(M_j'\tilde{z} + c_j)$ on $\mathrm{supp}(\gamma_j^{*\prime} X)$, so on the interior of that set $\bar{P}$ is C^1 with $D\bar{P}(\tilde{z}) = M_j DP_o(M_j'\tilde{z} + c_j)$ after using the symmetry of DP_o. Moreover $\bar{P}$ is the gradient of a C^1 surplus function (Lemma B.1), which is C^2 wherever $\bar{P}$ is C^1. Hence $D\bar{P}$ is symmetric there, and transposing gives $M_j DP_o = DP_o M_j'$ on $\mathrm{Int}\,(\mathrm{supp}(\gamma_{oj}' X))$. Steps 4 and 5 of Theorem 1 are algebra from this identity and Assumption 3(c), so $M_j = d_j \mathbf{I}_2$ with d_j real and nonzero. For the sign, monotonicity of $\bar{P}$ (Lemma B.1) makes $D\bar{P}$ positive semidefinite where it exists. The form $D\bar{P} = d_j DP_o(d_j \cdot + c_j)$ with DP_o positive definite rules out $d_j < 0$.

The cross-judge comparison uses only level identities, the injectivity of $\bar{P}$ on the sets where its Jacobian is positive definite (Lemma I.6, whose proof applies to $\bar{P}$ restricted to those open sets), Lemma I.5, and the normalizations. It yields $d_j \equiv d = 1$ and $c_j \equiv 0$, hence $\boldsymbol{\gamma}^* = \boldsymbol{\gamma}_o$. □

Proof of Theorem 2(i). The rate claim is Proposition B.1. For consistency of $\hat{\boldsymbol{\gamma}}$, fix $\epsilon > 0$. The estimated propensity satisfies $P_{\hat{\xi}} \in \mathcal{P} \subseteq \bar{\mathcal{P}}$ (standing conventions), so on the event $\{\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\| \geq \epsilon\}$ Lemma B.6 gives $\mathsf{L}(\hat{\theta}, \theta_o) \geq \delta(\epsilon) > 0$. Since $\mathsf{L}(\hat{\theta}, \theta_o) = o_{\mathrm{P}}(1)$, the probability of this event vanishes, i.e. $\hat{\boldsymbol{\gamma}} \to_p \boldsymbol{\gamma}_o$. □

Cells and notation for the curvature bound. A *cell* is a pair $c = (j, x_b)$ with $j \in \{1, \ldots, J\}$ and $x_b \in \mathrm{supp}(X_b)$. There are finitely many cells, each with probability $\pi_c = \mathbb{P}(J = j, X_b = x_b) \geq \underline{\pi} > 0$ by Assumption 4(a) and the definition of the support. On cell c, the index is affine in the continuous covariates, $\boldsymbol{\gamma}'W = \gamma_{j,1} + \gamma_{j,a}' X_a + \gamma_{j,b}' x_b$. Write $f_c(\cdot\,|\boldsymbol{\gamma})$ for the density of $\boldsymbol{\gamma}'W$ given the cell c (Lemma B.7), $f_{\boldsymbol{\gamma}} := \sum_c \pi_c f_c(\cdot\,|\boldsymbol{\gamma})$ for the index density, and $\boldsymbol{e}$ for the intercept stack of Section 3.1, so that $\boldsymbol{e}'W_i \equiv 1$. By Assumption 3(b) and continuity, there are a convex neighborhood $\mathcal{N}$ of $\boldsymbol{\gamma}_o$ in $\{\boldsymbol{\gamma} : \boldsymbol{e}'\boldsymbol{\gamma}^1 = \boldsymbol{e}'\boldsymbol{\gamma}^2 = 0\}$ and $\underline{\sigma} > 0$ with $\sigma_2(\gamma_{j,a}) \geq \underline{\sigma}$ for every j and $\boldsymbol{\gamma} \in \mathcal{N}$.

Lemma B.7 (cell index densities: existence, bound, L^1 continuity)**.** *Suppose Assumptions 3(b) and 4(a),(d) hold. Then, for every cell c and $\boldsymbol{\gamma} \in \mathcal{N}$, $\boldsymbol{\gamma}'W$ given c admits a density $f_c(\cdot\,|\boldsymbol{\gamma})$ on $\mathbb{R}^2$ with $\sup_{\boldsymbol{\gamma}\in\mathcal{N}} \|f_c(\cdot\,|\boldsymbol{\gamma})\|_\infty \leq \bar{f} < \infty$, and as $\boldsymbol{\gamma} \to \boldsymbol{\gamma}_o$,*

$$\int_{\mathbb{R}^2} |f_c(z\,|\boldsymbol{\gamma}) - f_c(z\,|\boldsymbol{\gamma}_o)| dz \to 0.$$

Proof. On cell c, $\boldsymbol{\gamma}'W = \gamma'_{j,a}X_a + b_{\boldsymbol{\gamma}}$, where $\gamma'_{j,a} \in \mathbb{R}^{2\times\dim(X_a)}$ has rank 2 and $b_{\boldsymbol{\gamma}} = \gamma_{j,1} + \gamma'_{j,b}x_b$. By the coarea formula, the pushforward of $p_{X_a\,|\,X_b}(\cdot\,|\,x_b)dx_a$ under $x_a \mapsto \gamma'_{j,a}x_a + b_{\boldsymbol{\gamma}}$ has the density

$$f_c(z\,|\,\boldsymbol{\gamma}) = \mathcal{J}(\boldsymbol{\gamma})^{-1}\int_{\ker\gamma'_{j,a}} p_{X_a\,|\,X_b}((\gamma'_{j,a})^+(z - b_{\boldsymbol{\gamma}}) + \nu\,|\,x_b)d\nu,$$

where $\mathcal{J}(\boldsymbol{\gamma}) = \sqrt{\det(\gamma'_{j,a}\gamma_{j,a})}$. The fiber is contained in a ball of radius R_W, the conditional density is bounded (continuous with compact support), and $\mathcal{J}(\boldsymbol{\gamma}) \geq \underline{\sigma}^2$ on $\mathcal{N}$, which gives the uniform bound $\bar{f}$. For the L^1 continuity, parametrize $\ker\gamma'_{j,a}$ by an orthonormal basis chosen continuously in $\boldsymbol{\gamma}$ on the (shrunken) neighborhood $\mathcal{N}$. The map $(z,\nu) \mapsto (\gamma'_{j,a})^+(z - b_{\boldsymbol{\gamma}}) + O_{\boldsymbol{\gamma}}\nu$ is affine with coefficients converging to those at $\boldsymbol{\gamma}_o$, and $\mathcal{J}(\boldsymbol{\gamma}) \to \mathcal{J}(\boldsymbol{\gamma}_o)$. Since composition of a fixed L^1 function with converging invertible affine maps is continuous in L^1 (continuity of translation and dilation in L^1, applied to the bounded, compactly supported $p_{X_a\,|\,X_b}$), integrating out ν over the bounded fiber and z over the bounded range yields the displayed convergence by Fubini and dominated convergence. □

Lemma B.8 (local regularity)**.** *Suppose Assumptions 4(a),(d) hold, and let $\{B_e\}_{e\in\mathcal{E}}$ be any finite collection of open balls with closures in $\mathrm{Int}(\mathcal{Z})$. There are $\eta > 0$, $\pi_0 > 0$, and a neighborhood $\mathcal{N}$ of $\boldsymbol{\gamma}_o$ such that, with the eroded support $\mathcal{Z}_\eta := \{z \in \mathcal{Z} : \mathrm{dist}(z, \partial\mathcal{Z}) \geq \eta\}$: (i) $\bar{\mathcal{Z}}_\eta \subset \mathrm{Int}(\mathcal{Z})$ and $B_e \subset \mathcal{Z}_\eta$ for every $e \in \mathcal{E}$; (ii) $\mathbb{P}(\boldsymbol{\gamma}'W \in \mathcal{Z}_\eta\,|\,c) \geq \pi_0$ for every cell c and every $\boldsymbol{\gamma} \in \mathcal{N}$; (iii) $\inf_{\mathcal{Z}_\eta} p_Z > 0$.*

Proof. For each cell c, $\mathbb{P}(\boldsymbol{\gamma}'_oW \in \mathrm{Int}(\mathcal{Z})\,|\,c) = 1$ since the index has a density and $\mathcal{Z}\setminus\mathrm{Int}(\mathcal{Z})$ has measure zero (compact support with dense interior), so $\mathbb{P}(\boldsymbol{\gamma}'_oW \in \mathcal{Z}_{2\eta}\,|\,c) \uparrow 1$ as $\eta \downarrow 0$. Choose η small enough that this probability exceeds $1/2$ for every cell and that (i) holds. With $R_W\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_o\| < \eta$ on $\mathcal{N}$, any realization with $\boldsymbol{\gamma}'_oW \in \mathcal{Z}_{2\eta}$ has $\boldsymbol{\gamma}'W \in \mathcal{Z}_\eta$, giving (ii) with $\pi_0 = 1/2$. Claim (iii) holds because p_Z is continuous and positive on the compact $\mathcal{Z} \supseteq \bar{\mathcal{Z}}_\eta$. □

Lemma B.9 (non-degeneracy of the index field)**.** *Suppose Assumption 3(c) and Assumption 4(b),(d) hold, and let $\mathcal{Z}_\eta$ be as in Lemma B.8 with η small. For $M \in \mathbb{R}^{2\times2}$ define $\Phi_M(z) := DP_o(z)M'z$ and let $\mathcal{G}$ be the class of gradient fields on $\mathcal{Z}_\eta$. There is $\underline{\lambda} > 0$ such that, for every M with $\mathrm{vec}(M) \perp \mathrm{vec}(\mathbf{I}_2)$,*

$$\inf_{g\in\mathcal{G}}\int_{\mathcal{Z}_\eta}\|\Phi_M(z) - g(z)\|^2dz \geq \underline{\lambda}\|M\|^2.$$

Proof. The Jacobian $DP_o = \nabla^2 G_o$ is symmetric, positive definite (Lemma I.3), and C^1 on $\bar{\mathcal{Z}}$ ($\lfloor k_o \rfloor \geq 2$). The infimum equals the squared $L^2(\mathcal{Z}_\eta)$ norm of the component of Φ_M orthogonal to all gradient fields, which is positive if and only if Φ_M violates the cross-partial equality somewhere on $\mathcal{Z}_\eta$. Because third derivatives of G_o commute, the cross-partial asymmetry of the field is

$$\partial_{z^1}\Phi_M^2 - \partial_{z^2}\Phi_M^1 = \langle S_M, DP_o(z)\rangle, \qquad S_M = \begin{pmatrix} -M_{21} & \frac{1}{2}(M_{11}-M_{22}) \\ \frac{1}{2}(M_{11}-M_{22}) & M_{12} \end{pmatrix},$$

with $\langle S, DP_o\rangle := S_{11}D_{11}G_o + 2S_{12}D_{12}G_o + S_{22}D_{22}G_o$. The map $M \mapsto S_M$ is linear with kernel $\operatorname{span}(\mathbf{I}_2)$ and $\|S_M\| \asymp |||M|||$ on $\operatorname{vec}(\mathbf{I}_2)^\perp$. For nonzero symmetric S, $\langle S, DP_o\rangle \not\equiv 0$ on $\operatorname{Int}(\mathcal{Z})$: if S is indefinite, $S = \frac{1}{2}(v_1 v_2' + v_2 v_1')$ with v_1, v_2 linearly independent and $\langle S, DP_o\rangle = v_1' DP_o v_2$, which is not identically zero by Assumption 3(c) applied on $\operatorname{Int}(\operatorname{supp}(\gamma_{oj}' X)) \subseteq \operatorname{Int}(\mathcal{Z})$; if S is definite, $\langle S, DP_o\rangle = \operatorname{tr}(SDP_o) \neq 0$; if $S = \pm uu'$, $\langle S, DP_o\rangle = \pm u' DP_o u \neq 0$, the last two by $DP_o \succ 0$. Hence $c_0 := \inf_{\|S\|=1} \int_{\operatorname{Int}(\mathcal{Z})} \langle S, DP_o\rangle^2 dz > 0$ by compactness of the unit sphere and continuity. Eroding to $\mathcal{Z}_\eta$ removes the interior strip $\operatorname{Int}(\mathcal{Z}) \setminus \mathcal{Z}_\eta$, whose Lebesgue measure vanishes as $\eta \downarrow 0$. Since $|\langle S, DP_o\rangle| \leq \|DP_o\|_\infty$, $\int_{\mathcal{Z}_\eta} \langle S, DP_o\rangle^2 \geq c_0 - \|DP_o\|_\infty^2 \operatorname{meas}(\operatorname{Int}(\mathcal{Z}) \setminus \mathcal{Z}_\eta) \geq c_0/2$ for η small, uniformly in $\|S\| = 1$. Finally $\underline{\lambda} := \min\{\inf_g \int_{\mathcal{Z}_\eta} \|\Phi_M - g\|^2 : \|\operatorname{vec}(M)\| = 1, \operatorname{vec}(M) \perp \operatorname{vec}(\mathbf{I}_2)\} > 0$ by compactness, since each value is positive by the cross-partial computation above. □

Lemma B.10 (overlap chaining)**.** *Suppose Assumptions 3(e) and 4(a),(d) hold, and suppose each judge's index $\gamma_{oj}' X_i$ admits a continuous density f_j. There exist a connected graph $\mathcal{G}^*$ on $\{1, \ldots, J\}$ and, for each of its edges (j, j'), a closed ball $B_{jj'} \subset \operatorname{Int}(\mathcal{Z})$ on which $\min\{f_j, f_{j'}\} \geq \underline{f} > 0$.*

Proof. The sets $O_j := \{f_j > 0\}$ are open by the assumed continuity, and $\operatorname{supp}(\gamma_{oj}' X) = \bar{O}_j$. Fix an edge (j, j') of the examiner-adjacency graph and let C be a connected component of the open overlap $\operatorname{Int}(\operatorname{supp}(\gamma_{oj}' X)) \cap \operatorname{Int}(\operatorname{supp}(\gamma_{oj'}' X))$, which is nonempty by Assumption 3(e). First, C meets both O_j and $O_{j'}$: C is an open subset of $\bar{O}_j$, and an open set disjoint from O_j cannot lie in $\bar{O}_j \setminus O_j$, the boundary of an open set having empty interior. Second, $C \subseteq \cup_l O_l$: at every $z \in C \subseteq \mathcal{Z}$ the mixture $p_Z(z) = \sum_l \lambda_l f_l(z)$ is positive, so some $f_l(z) > 0$. A connected set covered by open sets is chained: any two of its points are joined by a finite sequence of the covering sets with consecutive pairwise intersections nonempty in C. Applying this between a point of $C \cap O_j$ and a point of $C \cap O_{j'}$ yields judges $j = l_0, l_1, \ldots, l_m = j'$ with $O_{l_{s-1}} \cap O_{l_s} \cap C \neq \varnothing$ for every s. Each such

intersection is open and nonempty, and continuity supplies a closed ball inside it on which both densities exceed a positive constant. Collect these pairs (l_{s-1}, l_s), over all edges of the examiner-adjacency graph, as the edges of $\mathcal{G}^*$, with the corresponding balls, taking $\underline{f}$ as the minimum over the finitely many balls. Every edge of the examiner-adjacency graph connects its endpoints within $\mathcal{G}^*$, so connectedness of $\mathcal{G}^*$ follows from Assumption 3(e). □

Proposition B.2 (primitive sufficient conditions for Assumption 4(e))**.** *Suppose Assumptions 1, 3 and 4(a),(d) hold, and in addition: (i) each judge's index $\gamma_{oj}'X_i$ admits a continuous density; (ii) writing $X_{-1,i} = (X_{a,i}', X_{b,i}')'$ for the covariates excluding the intercept and $\gamma_{j,-1}$ conformably, there exist $\epsilon, \underline{v} > 0$ such that $\mathrm{Var}(v'X_{-1,i} \,|\, \gamma_j'X_i) \geq \underline{v}\|v\|^2$ almost surely, for every judge j and every (v, γ_j) with $\gamma_{j,-1}'v = 0$ and $\|\gamma_j - \gamma_{oj}\| < \epsilon$. Then Assumption 4(e) holds: there are $\delta' > 0$ and a neighborhood $\mathcal{N}$ of $\boldsymbol{\gamma}_o$ such that, for every $\boldsymbol{\gamma} \in \mathcal{N}$ and every $u \in \mathbb{R}^{\dim(W_i)}$ with $u \perp \mathrm{span}(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{e})$,*

$$\mathbb{E}\left[\mathrm{Var}(u'W \,|\, \boldsymbol{\gamma}'W)\right] \geq \delta'\|u\|^2.$$

Proof. Let $\mathcal{G}^*$ and the balls $\{B_{jj'}\}$ be as in Lemma B.10, whose hypotheses hold under (i). Fix $\boldsymbol{\gamma} \in \mathcal{N}$ and $u \perp \mathrm{span}(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{e})$. All constants below are uniform over both. Decompose each judge block as $u_j = (u_{j,1}, u_{j,-1})$ and $u_{j,-1} = \gamma_{j,-1}\alpha_j + \tilde{u}_j$ with $\gamma_{j,-1}'\tilde{u}_j = 0$. The maps $u \mapsto \alpha_j$ and $u \mapsto \tilde{u}_j$ are linear with coefficients continuous in $\boldsymbol{\gamma}$ on $\mathcal{N}$, by the rank condition and continuity. Since $\gamma_j'X = \gamma_{j,1} + \gamma_{j,-1}'X_{-1}$, on the event $\{J = j, \boldsymbol{\gamma}'W = z\}$,

$$u'W = u_j'X = \alpha_j'z + c_j + \tilde{u}_j'X_{-1}, \qquad c_j := u_{j,1} - \alpha_j'\gamma_{j,1},$$

and, because X is independent of J, the conditional variance decomposes over judges as

$$\mathbb{E}\left[\mathrm{Var}(u'W \,|\, Z)\right] = \underbrace{\sum_j \mathbb{E}\left[\mathbb{P}(J = j \,|\, Z)\,\mathrm{Var}(\tilde{u}_j'X_{-1} \,|\, \gamma_j'X = Z)\right]}_{\text{within}} + \underbrace{\mathbb{E}\left[\mathrm{Var}_J(a_J(Z) \,|\, Z)\right]}_{\text{across}}, \tag{B.4}$$

with $Z := \boldsymbol{\gamma}'W$, $a_j(z) := \alpha_j'z + c_j + \mu_j(z)$, and $\mu_j(z) := \mathbb{E}[\tilde{u}_j'X_{-1} \,|\, \gamma_j'X = z]$, so that $|\mu_j| \leq R_W\|\tilde{u}_j\|$.

Within. Hypothesis (ii) floors the conditional variance by $\underline{v}\|\tilde{u}_j\|^2$ for almost every z, uniformly over $\boldsymbol{\gamma} \in \mathcal{N}$, and $\mathbb{E}[\mathbb{P}(J = j \,|\, Z)] = \lambda_j$ by independence. Hence within $\geq \underline{v}\lambda \sum_j \|\tilde{u}_j\|^2 =: c_1\|\tilde{u}\|^2$.

Across. Let $f_j(\cdot \,|\, \boldsymbol{\gamma})$ denote the index density of judge j, the slice mixture of the cell

densities of Lemma B.7, which inherits the bound $\bar{f}$ and the L^1 continuity in $\boldsymbol{\gamma}$. By the pairwise variance identity, for each edge (j, j') of $\mathcal{G}^*$,

$$\begin{aligned}\text{across} &\geq \int_{B_{jj'}} \mathbb{P}(j \,|\, z)\mathbb{P}(j' \,|\, z)\,(a_j(z) - a_{j'}(z))^2\, f_{\boldsymbol{\gamma}}(z)dz \\ &= \lambda_j\lambda_{j'} \int_{B_{jj'}} \frac{f_j(z\,|\boldsymbol{\gamma})f_{j'}(z\,|\boldsymbol{\gamma})}{f_{\boldsymbol{\gamma}}(z)}\,(a_j - a_{j'})^2\, dz.\end{aligned}$$

Lemma B.10 floors $f_j(\cdot\,|\boldsymbol{\gamma}_o)$ and $f_{j'}(\cdot\,|\boldsymbol{\gamma}_o)$ on $B_{jj'}$, and the L^1 continuity gives, after shrinking $\mathcal{N}$, a set $S_{\boldsymbol{\gamma}} \subseteq B_{jj'}$ with $\text{meas}(S_{\boldsymbol{\gamma}}) \geq \frac{3}{4}\,\text{meas}(B_{jj'})$ on which both densities exceed half their floors, while $f_{\boldsymbol{\gamma}} \leq \bar{f}$. The difference $a_j - a_{j'} = g_{jj'} + (\mu_j - \mu_{j'})$ splits into the affine function $g_{jj'}(z) := (\alpha_j - \alpha_{j'})'z + (c_j - c_{j'})$ and a remainder bounded by $2R_W\|\tilde{u}\|$, so $(a_j - a_{j'})^2 \geq \frac{1}{2}g_{jj'}^2 - 4R_W^2\|\tilde{u}\|^2$, and the sublevel-strip bound for affine functions on $S_{\boldsymbol{\gamma}}$ gives $\int_{S_{\boldsymbol{\gamma}}} g_{jj'}^2 \geq c_B(\|\alpha_j - \alpha_{j'}\|^2 + |c_j - c_{j'}|^2)$. Combining over the edges,

$$\text{across} \geq c_2 \sum_{(j,j')} \left(\|\alpha_j - \alpha_{j'}\|^2 + |c_j - c_{j'}|^2\right) - C_3\|\tilde{u}\|^2.$$

Combining. Averaging the two bounds with weights adapted to C_3, (B.4) gives $\mathbb{E}[\text{Var}(u'W \,|\, Z)] \geq c_4\|T_{\boldsymbol{\gamma}}u\|^2$ with the linear map $T_{\boldsymbol{\gamma}}u := ((\tilde{u}_j)_j, (\alpha_j - \alpha_{j'}, c_j - c_{j'})_{(j,j')})$, whose coefficients are continuous in $\boldsymbol{\gamma}$. Its kernel is exactly $\text{span}(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{e})$: $\tilde{u}_j \equiv 0$ forces $u_{j,-1} = \gamma_{j,-1}\alpha_j$, and connectedness of $\mathcal{G}^*$ (Lemma B.10) forces $\alpha_j \equiv \alpha$ and $c_j \equiv c$, whence $u_{j,-1} = \gamma_{j,-1}\alpha$ and $u_{j,1} = \alpha'\gamma_{j,1} + c$, i.e. $u = \boldsymbol{\gamma}\alpha + c\boldsymbol{e}$. Positivity and uniformity follow from continuity of $(\boldsymbol{\gamma}, u) \mapsto \|T_{\boldsymbol{\gamma}}u\|^2$ on the compact set $\{\boldsymbol{\gamma} \in \bar{\mathcal{N}}, \|u\| = 1, u \perp \text{span}(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{e})\}$. □

Proposition B.3 (variation control)**.** *Suppose Assumptions 1, 3 and 4(a),(d),(e) hold, Assumption 4(f) holds, and $k_o \geq 2$. There are $\epsilon_0 > 0$ and $c > 0$ such that, for every $\boldsymbol{\gamma} \in \Gamma$ with $|||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| < \epsilon_0$,*

$$\inf_{P \in \bar{\mathcal{P}}} \mathsf{L}\left((\boldsymbol{\gamma}, P), \theta_o\right) \geq c|||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o|||.$$

Proof. Fix $\boldsymbol{\gamma} \in \Gamma \cap \mathcal{N}$, $P \in \bar{\mathcal{P}}$, and write $z := \boldsymbol{\gamma}'W$, $z_o := \boldsymbol{\gamma}_o'W$, $E := \boldsymbol{\gamma} - \boldsymbol{\gamma}_o$, and $\mathrm{P}_{\boldsymbol{\gamma}} := \boldsymbol{\gamma}(\boldsymbol{\gamma}'\boldsymbol{\gamma})^{-1}\boldsymbol{\gamma}'$.

Step 1: exact decomposition of the index shift. Regressing z_o on z,

$$z - z_o = E'W = Mz - r,\, M := \mathbf{I}_2 - \boldsymbol{\gamma}_o'\boldsymbol{\gamma}(\boldsymbol{\gamma}'\boldsymbol{\gamma})^{-1} = E'\boldsymbol{\gamma}(\boldsymbol{\gamma}'\boldsymbol{\gamma})^{-1},\, r := \boldsymbol{\gamma}_o'(I - \mathrm{P}_{\boldsymbol{\gamma}})W = -E_\perp'W,$$

with $E_\perp := (I - \mathrm{P}_{\boldsymbol{\gamma}})E$, using $\boldsymbol{\gamma}'(I - \mathrm{P}_{\boldsymbol{\gamma}}) = 0$. With $B := (\boldsymbol{\gamma}'\boldsymbol{\gamma})^{-1}\boldsymbol{\gamma}'E$, so that $E = \boldsymbol{\gamma}B + E_\perp$,

one has $M = B'$ exactly.

Step 2: the residual block is orthogonal to $\boldsymbol{e}$. Because $\boldsymbol{\gamma}, \boldsymbol{\gamma}_o \in \Gamma$, the columns of E satisfy $\boldsymbol{e}'E^t = 0$, and $\boldsymbol{e} \perp \mathrm{col}(\boldsymbol{\gamma})$ since $\boldsymbol{e}'\boldsymbol{\gamma} = 0$, so $\boldsymbol{e}'E_\perp^t = \boldsymbol{e}'E^t = 0$. Each column of $E_\perp$ therefore lies in $\mathrm{span}(\boldsymbol{\gamma}^1, \boldsymbol{\gamma}^2, \boldsymbol{e})^\perp$. Writing $m_\perp(z) := \mathbb{E}[E'_\perp W \,|\, z]$ and $\tilde{E} := E'_\perp W - m_\perp(z)$, we have $r = -E'_\perp W$, $r - \mathbb{E}[r \,|\, z] = -\tilde{E}$.

Step 3: two orthogonal blocks in L^2. By a first-order expansion ($P_o \in C^2$), $P_o(z) - P_o(z_o) = DP_o(z)(Mz - r) + R_2$ with $\|R_2\| \le C\|E\|^2$. Collect the z-measurable terms. Splitting $M = M_\perp + \tau \mathbf{I}_2$ into traceless and trace parts, the trace part is absorbed as a gradient, $\Phi_{\tau \mathbf{I}_2}(z) = \tau DP_o(z)z = \tau \nabla(z'P_o(z) - G_o(z))$. Hence

$$P(z) - P_o(z_o) = \underbrace{\left[\widetilde{Q}(z) + \Phi_{M_\perp}(z) + DP_o(z)m_\perp(z)\right]}_{=:g(z)} + DP_o(z)\tilde{E} + R_2,$$

where $\widetilde{Q} := (P - P_o) +$ (absorbed gradient fields) is a gradient field bounded by a constant C_b (the cap bounds $P - P_o$, and the absorbed fields are bounded on the compact $\bar{\mathcal{Z}}$ by $C\|E\|$-multiples of fixed functions). Since g is z-measurable and $\mathbb{E}[\tilde{E} \,|\, z] = 0$, the cross term vanishes and

$$\mathsf{L}\left((\boldsymbol{\gamma}, P), \theta_o\right) \ge \left(\|g\|_{L^2}^2 + \|DP_o\tilde{E}\|_{L^2}^2\right)^{1/2} - C\|E\|^2. \tag{B.5}$$

Step 4: orthogonal block. On all of $\bar{\mathcal{Z}}$, $\lambda_{\min}(DP_o) \ge c_H > 0$: the Jacobian DP_o is positive definite at every point by Lemma I.3, using that densities in $\mathcal{F}$ are everywhere positive, and it is continuous on the compact $\bar{\mathcal{Z}}$. Applying Assumption 4(e) to each unit column $E_\perp^t / \|E_\perp^t\|$, which is orthogonal to the columns of the running $\boldsymbol{\gamma}$ and to $\boldsymbol{e}$,

$$\|DP_o\tilde{E}\|_{L^2}^2 \ge c_H^2 \mathbb{E}\|\tilde{E}\|^2 = c_H^2 \sum_t \mathbb{E}\left[\mathrm{Var}(E_\perp^{t\prime} W \,|\, z)\right] \ge c_H^2 c_3 \|E_\perp\|^2 =: c_5 \|E_\perp\|^2.$$

Step 5: index block. By Assumption 4(f) there are $0 < \underline{p} \le \bar{p}$ and a neighborhood $\mathcal{N}$ of $\boldsymbol{\gamma}_o$ with $\underline{p} \le p_{\boldsymbol{\gamma}} \le \bar{p}$ on $\mathrm{supp}(\boldsymbol{\gamma}'W)$ for every $\boldsymbol{\gamma} \in \mathcal{N}$. We claim that a further shrinking of $\mathcal{N}$ gives $\mathcal{Z}_\eta \subseteq \mathrm{supp}(\boldsymbol{\gamma}'W)$ for every $\boldsymbol{\gamma} \in \mathcal{N}$. Write $\mathrm{supp}(W) = \cup_c (\{1\} \times \mathcal{C}_c \times \{x_{b,c}\})$ over the finitely many cells $c = (j, x_b)$, with $\mathcal{C}_c := \mathrm{supp}(X_a \,|\, X_b = x_{b,c})$, so that $\mathrm{supp}(\boldsymbol{\gamma}'W) = \cup_c T_{\boldsymbol{\gamma},c}(\mathcal{C}_c)$ with $T_{\boldsymbol{\gamma},c}(x_a) := \gamma_{j,1} + \gamma'_{j,a}x_a + \gamma'_{j,b}x_{b,c}$. By Assumption 4(f), every $z \in \bar{\mathcal{Z}}_\eta \subset \mathrm{Int}\,(\mathcal{Z})$ equals $T_{\boldsymbol{\gamma}_o,c}(x_z)$ for some cell c and some $x_z \in \mathrm{Int}\,(\mathcal{C}_c)$. Fix such z, c, x_z, and $\varrho_z > 0$ with $B(x_z, \varrho_z) \subseteq \mathcal{C}_c$. The linear part of $T_{\boldsymbol{\gamma},c}$ is $\gamma'_{j,a}$, whose smaller singular value is at least

$\underline{\sigma}$ once $\mathcal{N}$ is intersected with the neighborhood on which $\sigma_2(\gamma_{j,a}) \geq \underline{\sigma}$, so $T_{\boldsymbol{\gamma},c}(B(x_z, \varrho_z)) \supseteq B(T_{\boldsymbol{\gamma},c}(x_z), \underline{\sigma}\varrho_z)$ for every $\boldsymbol{\gamma} \in \mathcal{N}$. Since $\|T_{\boldsymbol{\gamma},c}(x_z) - z\| \leq R_W |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o|||$, the ball $B(z, \underline{\sigma}\varrho_z/2)$ lies in $T_{\boldsymbol{\gamma},c}(\mathcal{C}_c) \subseteq \operatorname{supp}(\boldsymbol{\gamma}' W)$ whenever $R_W |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| \leq \underline{\sigma}\varrho_z/2$. Compactness of $\bar{\mathcal{Z}}_\eta$ yields finitely many such balls covering it, and taking $\mathcal{N}$ inside the smallest of the corresponding neighborhoods of $\boldsymbol{\gamma}_o$ proves the claim. Hence $p_{\boldsymbol{\gamma}} \geq \underline{p}$ on $\mathcal{Z}_\eta$ throughout $\mathcal{N}$. The field $\widetilde{Q}$ is a gradient field in $L^2(\mathcal{Z}_\eta)$, so by the triangle inequality in L^2 and $\|DP_o m_\perp\|_{L^2} \leq C\|E_\perp\|$,

$$\begin{aligned}\|g\|_{L^2} &\geq \left(\int_{\mathcal{Z}_\eta} \|\Phi_{M_\perp}(z) + \widetilde{Q}(z)\|^2 p_{\boldsymbol{\gamma}}(z)dz\right)^{1/2} - C\|E_\perp\| \\ &\geq \left(\underline{p}\int_{\mathcal{Z}_\eta} \|\Phi_{M_\perp} + \widetilde{Q}\|^2 dz\right)^{1/2} - C\|E_\perp\|,\end{aligned}$$

and Lemma B.9, whose infimum runs over all gradient fields and is homogeneous of degree two, bounds the integral below by $\underline{\lambda}|||M_\perp|||^2$. Hence $\|g\|_{L^2} \geq c_6|||M_\perp||| - C\|E_\perp\|$ with $c_6 := (\underline{p}\underline{\lambda})^{1/2}$, uniformly over $\boldsymbol{\gamma} \in \mathcal{N}$.

Step 6: norm accounting and combination. The normalization $|||\boldsymbol{\gamma}||| = |||\boldsymbol{\gamma}_o||| = 1$ gives $\operatorname{tr}(\boldsymbol{\gamma}'\boldsymbol{\gamma}B) = \operatorname{tr}(\boldsymbol{\gamma}'E) = \frac{1}{2}|||E|||^2$. Writing $B = B_0 + \tau \mathbf{I}_2$ with $\operatorname{tr}(\boldsymbol{\gamma}'\boldsymbol{\gamma}B_0) = 0$ yields $|\tau| = \frac{1}{2}\|E\|^2 / \operatorname{tr}(\boldsymbol{\gamma}'\boldsymbol{\gamma}) = \frac{1}{2}\|E\|^2$. On the subspace $\{\operatorname{tr}(\boldsymbol{\gamma}'\boldsymbol{\gamma}B_0) = 0\}$ the linear map $B_0 \mapsto (B_0')_{\text{traceless}}$ is injective (its kernel $B_0 \propto \mathbf{I}_2$ meets the subspace only at 0, since $\operatorname{tr}(\boldsymbol{\gamma}'\boldsymbol{\gamma}) = 1 \neq 0$), so $\|(B_0')_{\text{traceless}}\| \geq c_7\|B_0\|$ with c_7 uniform over $\mathcal{N}$. Hence, with $M = B'$,

$$\|M_\perp\| \geq c_7\|B\| - C\|E\|^2.$$

Also $\|E\|^2 = \|\boldsymbol{\gamma}B\|^2 + \|E_\perp\|^2 \leq C(\|B\|^2 + \|E_\perp\|^2)$ by the uniform positive definiteness of $\boldsymbol{\gamma}'\boldsymbol{\gamma}$ on $\mathcal{N}$. Fix Λ large. If $\|B\| \leq \Lambda\|E_\perp\|$, then (B.5) and Step 4 give $\mathsf{L} \geq c_5^{1/2}\|E_\perp\| - C\|E\|^2 \gtrsim \|E\| - C\|E\|^2$. If $\|B\| > \Lambda\|E_\perp\|$, then Steps 5–6 give $\mathsf{L} \geq c_6(c_7 - C/\Lambda)\|B\| - C\|E\|^2 \gtrsim \|E\| - C\|E\|^2$ for Λ large. Choosing ϵ_0 small absorbs the quadratic terms and proves the claim. □

Proof of Theorem 2(ii), first claim. By Theorem 2(i), $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| < \epsilon_0$ with probability tending to one. On that event, since $P_{\hat{\xi}} \in \mathcal{P} \subseteq \bar{\mathcal{P}}$, Proposition B.3 gives $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| \leq c^{-1}\mathsf{L}(\hat{\theta}, \theta_o)$, and the right side is $O_{\mathrm{P}}(\rho_N)$ by Proposition B.1. □

Proposition B.4 (uniform rate of the estimated propensity, interior version)**.** *Suppose the conditions of Theorem 2(ii) hold, and let* $\mathcal{Z}_N := \{z \in \mathcal{Z} : \operatorname{dist}(z, \partial\mathcal{Z}) \geq CK_N^{-1}\}$*, where* C *is large enough that every B-spline basis function active at a point of* $\mathcal{Z}_N$ *has support*

contained in $\mathcal{Z}$, a requirement met by a constant depending only on q and the mesh ratio. Then, for $t \in \{1, 2\}$,

$$\|P^t_{\hat{\xi}} - P^t_o\|_{L^\infty(\mathcal{Z}_N)} = O_{\mathrm{P}}\left(K_N \rho_N\right).$$

Proof. First remove the index error: by Lemma B.3(i) and Theorem 2(ii), $(\mathbb{E}\sum_t (P^t_{\hat{\xi}}(\boldsymbol{\gamma}_o' W) - P^t_o(\boldsymbol{\gamma}_o' W))^2)^{1/2} \le \mathsf{L}(\hat{\theta}, \theta_o) + 2MR_W \|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\| = O_{\mathrm{P}}(\rho_N)$. Since $p_Z \ge \underline{p}_{\mathcal{Z}} := \min_{\mathcal{Z}} p_Z > 0$ (Assumption 4(d)), it follows that $\|P^t_{\hat{\xi}} - P^t_o\|_{L^2(\mathcal{Z}, dz)} = O_{\mathrm{P}}(\rho_N)$. Interpose the approximant of Lemma B.4: $s := P^t_{\hat{\xi}} - P^t_{\xi^\circ}$ is a spline of order $(q-1, q)$ on the mesh with $\|s\|_{L^2(\mathcal{Z}, dz)} = O_{\mathrm{P}}(\rho_N + K_N^{-k_o}) = O_{\mathrm{P}}(\rho_N)$. For $z \in \mathcal{Z}_N$, every basis function active at z has support contained in $\mathcal{Z}$, so the local stability of the B-spline basis (De Boor, 1978, Ch. XI) gives $|s(z)| \le C\|(\text{active coefficients})\|_{\ell^\infty} \le C\|(\text{active coefficients})\|_{\ell^2} \le Ch^{-1}\|s\|_{L^2(\mathcal{Z}, dz)}$, whence $\|s\|_{L^\infty(\mathcal{Z}_N)} = O_{\mathrm{P}}(K_N \rho_N)$. The triangle inequality with $\|P^t_{\xi^\circ} - P^t_o\|_\infty \lesssim K_N^{-k_o}$ completes the proof. □

C Asymptotic Normality and Efficiency

This section proves Theorems 3 and C.1. The argument for Theorem 3 has four parts. Part I constructs the least favorable direction: it is characterized as the solution of an elliptic variational problem, inherits smoothness from the primitives of the model (Lemma C.2), and is approximated in the sieve with the rates and uniform bounds that the expansion requires (Lemma C.3). The nondegeneracy of the efficient information is Lemma C.1. Part II collects the localization inputs: the maximal inequality (Lemma C.4), uniform rates on the core (Lemma C.5), the strictly feasible anchor (Lemma C.7), and the exact identity for the population score (Lemma C.8). Part III derives the expansion: the criterion is compared along feasible perturbation paths that join $\hat{\theta}$ to anchor points shifted along the least favorable direction, and the resulting pair of one-sided inequalities pins the Fisher pairing of $\hat{\theta} - \theta_o$ with that direction (Lemmas C.9–C.12). Part IV contains the central limit theorem for the direction score (Lemma C.13) and assembles the proof. Section C.1 states Theorem C.1, on regular functionals of $(\boldsymbol{\gamma}, P)$, and proves it by extending the expansion to directions in the propensity alone (Lemmas C.14–C.16).

Notation. Write $\omega_i = (T_i, W_i)$, $Z = \boldsymbol{\gamma}_o' W$, and, for $\xi \in \Xi(K_N)$, $g^t := g^t_{K_N} = \partial_{z^t} b_{K_N}$ as in the conventions above, with $g^0 = -(g^1 + g^2)$. The index score of $l(\omega, \boldsymbol{\gamma}, P)$ and the

nuisance score in a coefficient direction $a \in \mathbb{R}^{d_{K_N}}$ are

$$s_{\boldsymbol{\gamma}}(\omega, \boldsymbol{\gamma}, P) = \sum_t \frac{T^t}{P^t(\boldsymbol{\gamma}'W)} DP^t(\boldsymbol{\gamma}'W) \otimes W,$$
$$s_\xi[a](\omega, \boldsymbol{\gamma}, P) = \sum_t \frac{T^t}{P^t(\boldsymbol{\gamma}'W)} g^t(\boldsymbol{\gamma}'W)'a.$$

For $\theta = (\boldsymbol{\gamma}, P)$, the metric of Section 5.1 is

$$\mathsf{L}(\theta, \theta_o) = \|P(\boldsymbol{\gamma}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot)\|_{L^2(p_W)}.$$

For triples of fields $u = (u^0, u^1, u^2)$ and $\tilde{u} = (\tilde{u}^0, \tilde{u}^1, \tilde{u}^2)$ on $\mathrm{supp}(W)$ with $\sum_t u^t = \sum_t \tilde{u}^t = 0$, the Fisher inner product is

$$\langle u, \tilde{u} \rangle_{P_o} := \mathbb{E} \sum_t u^t(W)\tilde{u}^t(W)/P_o^t(Z),$$

with the induced norm $\|\cdot\|_{P_o}$. A field of z is identified with the field of w obtained by evaluation at $z = \boldsymbol{\gamma}_o'w$. A triple u with $\sum_t u^t = 0$ is identified with the $\mathbb{R}^2$-valued field (u^1, u^2), which determines $u^0 = -(u^1 + u^2)$. For $\mathbb{R}^2$-valued fields u and $\tilde{u}$ of z, eliminating the 0 component gives

$$\langle u, \tilde{u} \rangle_{P_o} = \mathbb{E}[u(Z)'\mathbb{W}(Z)\tilde{u}(Z)] = \int_{\mathcal{Z}} u(z)'\mathbb{W}(z)\tilde{u}(z)p_Z(z)dz,$$

where

$$\mathbb{W}(z) := \mathrm{diag}\left(1/P_o^1(z), 1/P_o^2(z)\right) + (1/P_o^0(z))\mathbf{1}\mathbf{1}'.$$

The matrix $\mathbb{W}(z)$ is symmetric with eigenvalues in $[1, 3/\kappa]$ on $\bar{\mathcal{Z}}$, since $\kappa \leq P_o^t \leq 1 - \kappa$ makes its smallest eigenvalue at least $\min(1/P_o^1, 1/P_o^2) \geq 1$ and its largest at most $\max(1/P_o^1, 1/P_o^2) + 2/P_o^0 \leq 3/\kappa$. As in Lemma I.3, G denotes a social surplus function, with propensity $P = \nabla G$, and G_o is the true one. A surplus perturbation is written δG, and it moves the propensity by $\nabla \delta G$. The nuisance tangent space is the $\|\cdot\|_{P_o}$-closure

$$\mathcal{T} := \overline{\{\nabla \delta G : \delta G \in C^\infty(\bar{\mathcal{Z}})\}},$$

where each $\nabla \delta G$ perturbs the propensity by moving the surplus at fixed $\boldsymbol{\gamma}_o$, and Π_∞

denotes the projection onto $\mathcal{T}$ in $\langle\cdot,\cdot\rangle_{P_o}$. For a matrix $V \in \mathbb{R}^{\dim(W)\times 2}$, let

$$\phi_V(w) := DP_o(\boldsymbol{\gamma}_o' w)V'w, \qquad f_V(z) := DP_o(z)V'm_W(z),$$
$$h_V^{\mathrm{lf}} := \Pi_\infty f_V, \qquad \phi_V^{\mathrm{eff}} := \phi_V - h_V^{\mathrm{lf}},$$

where $m_W(z) := \mathbb{E}[W \,|\, Z = z]$. When V is the matrix form of a direction $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, the same objects are written ϕ_v, f_v, h_v^{lf} and ϕ_v^{eff}. The field h_v^{lf} is the least favorable direction and ϕ_v^{eff} the efficient image, with $v' I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v = \|\phi_v^{\mathrm{eff}}\|^2_{P_o}$. For every $\mathbb{R}^2$-valued field u of z,

$$\langle u, \phi_V\rangle_{P_o} = \langle u, f_V\rangle_{P_o},$$
$$\langle u, \phi_V\rangle_{P_o} = \langle u, h_V^{\mathrm{lf}}\rangle_{P_o} \quad \text{if } u \in \mathcal{T},$$

the first by conditioning on Z and the second because $h_V^{\mathrm{lf}} = \Pi_\infty f_V$. The field f_V thus represents the linear functional $u \mapsto \langle u, \phi_V\rangle_{P_o}$ on all $\mathbb{R}^2$-valued fields, and h_V^{lf} represents it on $\mathcal{T}$. Section C.1 uses the same letters for the representers f_Λ and h_Λ of the propensity derivative of a regular functional Λ. A surplus perturbation whose propensity perturbation is h_v^{lf} or h_Λ is written δG_v or δG_Λ (Lemmas C.2 and C.14).

Neighborhoods and rates. Let

$$\mathcal{N} := \{\boldsymbol{\gamma} : \boldsymbol{e}'\boldsymbol{\gamma}^1 = \boldsymbol{e}'\boldsymbol{\gamma}^2 = 0, |||\boldsymbol{\gamma}||| \le 1, |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| \le C\rho_N\},$$
$$\mathcal{U}_N := \{(\boldsymbol{\gamma}, P_\xi) : \boldsymbol{\gamma} \in \mathcal{N},\ \xi \in \Xi(K_N),\ \mathsf{L}((\boldsymbol{\gamma}, P_\xi), \theta_o) \le C_U\rho_N\}.$$

By Theorem 2(ii) and Proposition B.1, $\hat\theta \in \mathcal{U}_N$ with probability tending to one, for C_U large enough. Set $r_N := \sqrt{K_N^2 \log N/N}$. Throughout, Assumption 5(a) gives $K_N^9(\log N)^9/N \to 0$, hence $K_N^6(\log N)^2/N \to 0$, and $K_N^{-k_o} = O(r_N)$, hence $\rho_N = O(r_N)$, $K_N\rho_N \to 0$ and $\sqrt{N}\rho_N^2 \to 0$. Also $\sqrt{N}\rho_N K_N^{-(1+\alpha)} = O(K_N^{-\alpha}\sqrt{\log N}) \to 0$, since $N = O(K_N^{2k_o+2}\log N)$ makes K_N grow polynomially in N. For a bounded open set $U \subseteq \mathbb{R}^2$ with Lipschitz boundary, $H^1(U)$ denotes the Sobolev space of functions $\varphi \in L^2(U)$ whose weak gradient $\nabla\varphi$ lies in $L^2(U)$, with norm $(\|\varphi\|^2_{L^2(U)} + \|\nabla\varphi\|^2_{L^2(U)})^{1/2}$; $C^\alpha := C^{0,\alpha}$; and $C^{k,\alpha}(\partial\mathcal{Z})$ is the space of functions on $\partial\mathcal{Z}$ that are of class $C^{k,\alpha}$ in the local graph coordinates of the boundary. The class $C^{k,\alpha}$ is equipped with its standard norm $\|\cdot\|_{C^{k,\alpha}}$.

Cell geometry and core stability. On the mesh of width $h_N \asymp K_N^{-1}$, a cell Q is *fully covered* if $Q \subseteq \mathcal{Z}$ up to a set of measure zero, *empty* if $|Q \cap \mathcal{Z}| = 0$, and a *boundary cell* otherwise. The *core* $\mathcal{A}$ consists of the fully covered cells whose mesh neighbors are also

fully covered, so $\operatorname{dist}(\Omega_{\mathcal{A}}, \mathcal{Z}^c) \geq h_N$. The *boundary layer* $\mathcal{W}$ collects the boundary cells and the outer layer of fully covered cells. The sets $\Omega_{\mathcal{A}}, \Omega_{\mathcal{W}}$ are the corresponding regions. By Lemma C.10(iii), $|\Omega_{\mathcal{W}}| = O(h_N)$, so $\mathbb{P}_{\boldsymbol{\gamma}_o}(\Omega_{\mathcal{W}}) \leq \bar{p}|\Omega_{\mathcal{W}}| = O(h_N)$, where $\mathbb{P}_{\boldsymbol{\gamma}}$ denotes the distribution of $\boldsymbol{\gamma}'W$, with density $p_{\boldsymbol{\gamma}} \in [\underline{p}, \bar{p}]$ on $\operatorname{supp}(\boldsymbol{\gamma}'W)$ for $\boldsymbol{\gamma}$ near $\boldsymbol{\gamma}_o$. For a cell Q, $\mathfrak{f}_Q := |Q \cap \mathcal{Z}|/|Q|$ is its covered fraction. Basis functions whose support meets $\mathcal{A}$ are called *interior*. The remaining ones, supported in $\Omega_{\mathcal{W}} \cup \Omega_{\mathcal{V}}$ with $\Omega_{\mathcal{V}}$ the empty region, are called *boundary* basis functions. Since W is bounded, $\operatorname{supp}(\boldsymbol{\gamma}'W)$ lies within $O(\rho_N)$ of $\mathcal{Z}$ for $\boldsymbol{\gamma} \in \mathcal{N}$, and $\rho_N = o(h_N)$ under Assumption 5(a), so $\Omega_{\mathcal{A}} \subseteq \operatorname{supp}(\boldsymbol{\gamma}'W)$ and $p_{\boldsymbol{\gamma}} \in [\underline{p}, \bar{p}]$ on $\Omega_{\mathcal{A}}$ for every $\boldsymbol{\gamma} \in \mathcal{N}$.

Lemma C.1 (nondegeneracy of the efficient information)**.** *Under Assumptions 1, 3 and 4(a), (b), (d) and (e), $v' I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v > 0$ for every nonzero $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$.*

Proof. Write $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ as a matrix $V \in \mathbb{R}^{\dim(W)\times 2}$, so that the path $\boldsymbol{\gamma}_o + \vartheta V$ shifts the index by $\vartheta V'W$. By construction $v' I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v = \|v's_{\boldsymbol{\gamma}} - \Pi_\infty(v's_{\boldsymbol{\gamma}})\|^2_{P_o}$, the squared Fisher distance from the index score in direction v to the nuisance tangent $\mathcal{T}$, and it vanishes if and only if $v's_{\boldsymbol{\gamma}} \in \mathcal{T}$. Differentiating the propensities along the path, the index score in direction v is carried by the field $w \mapsto DP_o(\boldsymbol{\gamma}_o' w)V'w$, while $\mathcal{T}$ is the closed span of the propensity perturbations $\nabla \delta G$ obtained by moving the surplus at fixed $\boldsymbol{\gamma}_o$, each a gradient field in z alone. Membership $v's_{\boldsymbol{\gamma}} \in \mathcal{T}$ therefore requires two things of $DP_o(\boldsymbol{\gamma}_o' W)V'W$: that it depend on W only through $Z = \boldsymbol{\gamma}_o' W$, and that as a field in z it be a gradient.

Decompose $V = \boldsymbol{\gamma}_o A + V_\perp$ with $A \in \mathbb{R}^{2\times 2}$ and the columns of $V_\perp$ orthogonal to $\boldsymbol{\gamma}_o^1$, $\boldsymbol{\gamma}_o^2$ and $\boldsymbol{e}$, which is the decomposition used in Step 1 of Proposition B.3. The two blocks are orthogonal in $\langle \cdot, \cdot \rangle_{P_o}$, since Φ_A is Z-measurable while $DP_o(Z)V_\perp' W$ has conditional mean zero given Z, so it suffices to bound each. For $V_\perp$, Assumption 4(e) gives $\mathbb{E}[\operatorname{Var}(V_\perp^{t\prime} W \,|\, Z)] \geq c_3 \|V_\perp^t\|^2$ for each column, so the part of $DP_o(Z)V_\perp' W$ that is not Z-measurable has Fisher norm bounded below by $c_H^2 c_3 |||V_\perp|||^2$ with $c_H := \inf_{\bar{\mathcal{Z}}} \lambda_{\min}(DP_o) > 0$ (Lemma I.3), and no element of $\mathcal{T}$ can match it. For $\boldsymbol{\gamma}_o A$, the induced field is $z \mapsto DP_o(z)A'z = \Phi_A(z)$ in the notation of Lemma B.9, which is Z-measurable, so only the gradient requirement binds. Lemma B.9 gives $\inf_{g \in \mathcal{G}} \int_{\mathcal{Z}_\eta} \|\Phi_A - g\|^2 \geq \underline{\lambda} |||A|||^2$ whenever $\operatorname{vec}(A) \perp \operatorname{vec}(\mathbf{I}_2)$, the infimum running over gradient fields, and $\mathcal{T}$ restricted to $\mathcal{Z}_\eta$ consists of gradient fields. The Lebesgue bound of Lemma B.9 transfers to $\langle \cdot, \cdot \rangle_{P_o}$ because p_Z is bounded away from zero on $\mathcal{Z}_\eta$ by Lemma B.8(iii) and $P_o^t \leq 1$. Hence Φ_A is matchable within $\mathcal{T}$ only when $A \propto \mathbf{I}_2$, that is only when $V \propto \boldsymbol{\gamma}_o$.

Collecting the two blocks, $v's_{\boldsymbol{\gamma}} \in \mathcal{T}$ forces $V_\perp = 0$ and $A \propto \mathbf{I}_2$, so $v \in \operatorname{span}(\operatorname{vec}(\boldsymbol{\gamma}_o)) \oplus \operatorname{span}\{\boldsymbol{e} \otimes u : u \in \mathbb{R}^2\}$, the radial direction together with the per-judge intercept-mean

directions. Both families are annihilated by $\Pi_{\boldsymbol{\gamma}_o}$ and lie outside $\mathcal{T}_{\boldsymbol{\gamma}_o}$. For nonzero $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ the residual is therefore nonzero, and $v' I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v > 0$. □

Lemma C.2 (the least favorable direction as an elliptic problem)**.**

(i) $\mathcal{T} = \{\nabla \delta G : \delta G \in H^1(\operatorname{Int} \mathcal{Z})\}$.

(ii) For every $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ *there is* $\delta G_v \in H^1(\operatorname{Int} \mathcal{Z})$*, unique up to an additive constant, such that* $h_v^{\text{lf}} = \nabla \delta G_v$ *and*

$$\int_{\mathcal{Z}} (\nabla \delta G_v - f_v)' \mathbb{W} \nabla \varphi \, p_Z dz = 0 \qquad \text{for all } \varphi \in H^1(\operatorname{Int} \mathcal{Z}). \tag{C.1}$$

(iii) Under Assumption 5(b), $\delta G_v \in C^{2,\alpha}(\mathcal{Z})$ *and* $\sup\{\|\delta G_v\|_{C^{2,\alpha}(\mathcal{Z})} : v \in \mathcal{T}_{\boldsymbol{\gamma}_o}, |||V||| = 1\} < \infty$.

Proof. (i): Smooth functions are dense in $H^1(\operatorname{Int} \mathcal{Z})$, and the set $\{\nabla \delta G : \delta G \in H^1\}$ is closed in the weighted L^2: if $\nabla \varphi_n \to u$, the Poincaré inequality on the connected Lipschitz domain $\operatorname{Int} \mathcal{Z}$ (Evans, 2010, Theorem 1, Section 5.8.1) makes the centered φ_n bounded in H^1, and its weak limit φ has $\nabla \varphi = u$. Hence $\mathcal{T} = \{\nabla \delta G : \delta G \in H^1(\operatorname{Int} \mathcal{Z})\}$.

(ii): Conditioning on Z gives $\langle \nabla \varphi, \phi_v \rangle_{P_o} = \int_{\mathcal{Z}} \nabla \varphi' \mathbb{W} f_v \, p_Z dz$, so projecting ϕ_v onto $\mathcal{T}$ is the same as projecting f_v. By (i), the projection h_v^{lf} minimizes $\int (\nabla \delta G - f_v)' \mathbb{W} (\nabla \delta G - f_v) p_Z$ over $\delta G \in H^1(\operatorname{Int} \mathcal{Z})$. The minimizer exists and is unique up to a constant by the Lax–Milgram theorem (Evans, 2010, Theorem 1, Section 6.2.1) on $H^1/\mathbb{R}$ with the uniformly elliptic form $\int \nabla \delta G' \mathbb{W} \nabla \varphi \, p_Z$, and (C.1) is its first-order condition.[9]

(iii): Equation (C.1) is the weak form of the divergence-form equation $\operatorname{div}(p_Z \mathbb{W}(\nabla \delta G_v - f_v)) = 0$ on $\operatorname{Int} \mathcal{Z}$ with the boundary condition $\mathbf{n}' p_Z \mathbb{W}(\nabla \delta G_v - f_v) = 0$ on $\partial \mathcal{Z}$, where $\mathbf{n}$ is the outward unit normal. The boundary condition is not imposed but implied, because the test functions φ are unrestricted on $\partial \mathcal{Z}$. It states that the weighted error $p_Z \mathbb{W}(\nabla \delta G_v - f_v)$ has no component along the outward normal. Since $\mathbb{W}$ is positive definite, the condition prescribes the derivative of δG_v in the outward-pointing direction $\mathbb{W}\mathbf{n}$, a condition of Neumann type. Gilbarg and Trudinger (2001, Section 6.7) treat such conditions under the name oblique derivative problem. Under Assumption 5(b) the coefficient matrix $p_Z \mathbb{W}$ is of class $C^{1,\alpha}(\mathcal{Z})$, since $p_Z \in C^{1,\alpha}$ and $P_o \in C^{k_o}$ with $\lfloor k_o \rfloor \geq 3$; the inhomogeneous term

[9] Here $H^1/\mathbb{R}$ denotes the quotient of $H^1(\operatorname{Int} \mathcal{Z})$ by the constant functions, whose elements are the classes $\{\delta G + c : c \in \mathbb{R}\}$ with $\delta G \in H^1(\operatorname{Int} \mathcal{Z})$. By the Poincaré inequality, the quotient norm is equivalent to $\|\nabla \delta G\|_{L^2(\operatorname{Int} \mathcal{Z})}$.

$f_v = DP_o V' m_W$ is of class $C^{1,\alpha}$; and $\partial\mathcal{Z}$ is of class $C^{2,\alpha}$. The compatibility condition $\int_{\mathcal{Z}} \operatorname{div}(p_Z \mathbb{W} f_v) dz = \int_{\partial\mathcal{Z}} \mathbf{n}' p_Z \mathbb{W} f_v ds$ holds by the divergence theorem, and the problem has a classical solution in $C^{2,\alpha}(\mathcal{Z})$, unique up to an additive constant, by the following functional-analytic argument.

Write $Lu := \operatorname{div}(p_Z \mathbb{W} \nabla u)$, $Nu := \mathbf{n}' p_Z \mathbb{W} \nabla u$, where L and N denote partial differential operators throughout this proof, $F := \operatorname{div}(p_Z \mathbb{W} f_v)$ and $\mu_v := \mathbf{n}' p_Z \mathbb{W} f_v$, so that the problem reads $Lu = F$ on $\operatorname{Int}\mathcal{Z}$ and $Nu = \mu_v$ on $\partial\mathcal{Z}$. Consider the auxiliary problem $Lu - u = g$ on $\operatorname{Int}\mathcal{Z}$, $Nu + u = \psi$ on $\partial\mathcal{Z}$, for $g \in C^\alpha(\mathcal{Z})$ and $\psi \in C^{1,\alpha}(\partial\mathcal{Z})$. The operator $L - \mathrm{id}$ is strictly elliptic with coefficients $p_Z \mathbb{W} \in C^{1,\alpha}(\mathcal{Z})$, first-order coefficients $\operatorname{div}(p_Z \mathbb{W}) \in C^\alpha(\mathcal{Z})$ and zero-order coefficient $-1 \le 0$. The boundary operator can be represented as $N + \mathrm{id} : u \mapsto \mathbf{b}' \nabla u + u$ with $\mathbf{b} := p_Z \mathbb{W} \mathbf{n} \in C^{1,\alpha}(\partial\mathcal{Z})$ and $\mathbf{b}'\mathbf{n} = p_Z \mathbf{n}' \mathbb{W} \mathbf{n} \ge \underline{p} > 0$ on $\partial\mathcal{Z}$, since $\mathbb{W} \ge \mathbf{I}$. The domain $\mathcal{Z}$ is $C^{2,\alpha}$. These are the hypotheses of Gilbarg and Trudinger (2001, Theorem 6.31), which gives a unique solution $u = S(g,\psi) \in C^{2,\alpha}(\mathcal{Z})$, and the estimate (6.80) in its proof gives $\|S(g,\psi)\|_{C^{2,\alpha}(\mathcal{Z})} \le C(\|g\|_{C^\alpha(\mathcal{Z})} + \|\psi\|_{C^{1,\alpha}(\partial\mathcal{Z})})$. A function $u \in C^{2,\alpha}(\mathcal{Z})$ solves $Lu = F$, $Nu = \mu_v$ if and only if $u = S(F - u, \mu_v + u|_{\partial\mathcal{Z}})$, that is, $u - S_0 u = S(F, \mu_v)$ with $S_0 u := S(-u, u|_{\partial\mathcal{Z}})$. The map S_0 is linear and bounded from $C^{1,\alpha}(\mathcal{Z})$ to $C^{2,\alpha}(\mathcal{Z})$ as shown above, hence compact as an operator on $C^{1,\alpha}(\mathcal{Z})$ as a consequence of the Arzela–Ascoli theorem, and thus the Fredholm alternative applies to $\mathrm{id} - S_0$ on $C^{1,\alpha}(\mathcal{Z})$ (Conway, 1990). Its kernel consists of the solutions of $Lu = 0$, $Nu = 0$ by construction. For such u, multiplying the equation by u and integrating by parts gives $\int_{\mathcal{Z}} \nabla u' p_Z \mathbb{W} \nabla u \, dz = 0$, so u is constant, and the kernel is one-dimensional. The range of $\mathrm{id} - S_0$ is therefore a closed subspace of codimension one by the Fredholm alternative. The data (F, μ_v) for which the problem is solvable are those with $S(F, \mu_v)$ in that range, a subspace of the data space of codimension at most one since $(F, \mu_v) \mapsto S(F, \mu_v)$ is linear. Conversely, every such datum satisfies the compatibility condition $\int_{\mathcal{Z}} F \, dz = \int_{\partial\mathcal{Z}} \mu_v \, ds$ by the divergence theorem, which itself defines a subspace of codimension one, so the two subspaces coincide and the problem is solvable for any data at hand. The solution so obtained satisfies the variational equation, hence coincides with δG_v up to a constant, which shows $\delta G_v \in C^{2,\alpha}(\mathcal{Z})$. For the bound in (iii), the solution with $\int_{\mathcal{Z}} \delta G_v = 0$ is the image of $S(F, \mu_v)$ under the bounded inverse of $\mathrm{id} - S_0$ restricted to its range[10] and the centered functions, so $\|\delta G_v\|_{C^{1,\alpha}(\mathcal{Z})} \le C\|S(F,\mu_v)\|_{C^{1,\alpha}} \le C(\|F\|_{C^\alpha} + \|\mu_v\|_{C^{1,\alpha}}) \le C\|f_v\|_{C^{1,\alpha}(\mathcal{Z})}$.

[10] We consider the quotient map $\widetilde{(\mathrm{id} - S_0)} : C^{1,\alpha}(\mathcal{Z})/\ker(\mathrm{id} - S_0) \to \operatorname{range}(\mathrm{id} - S_0)$ induced by $\mathrm{id} - S_0$ and its inverse, which is a bounded linear map by the inverse mapping theorem (Conway, 1990). Since $\ker(\mathrm{id} - S_0)$ comprises the constant functions, fixing $\int_{\mathcal{Z}} \delta G_v$ at zero resolves the ambiguity.

This implies that $\|\delta G_v\|_{C^{2,\alpha}} \leq \|S_0 \delta G_v\|_{C^{2,\alpha}} + \|S(F, \mu_v)\|_{C^{2,\alpha}} \leq C(\|\delta G_v\|_{C^{1,\alpha}} + \|F\|_{C^\alpha} + \|\mu_v\|_{C^{1,\alpha}}) \leq C\|f_v\|_{C^{1,\alpha}(\mathcal{Z})}$. The map $v \mapsto f_v$ is linear with $\|f_v\|_{C^{1,\alpha}} \leq C|||V|||$. □

The proof argument follows Shen (1997, Section 3), whose expansion of a sieve maximum likelihood estimator along a least favorable direction takes two inputs as given: a direction that the sieve approximates at a rate faster than the nonparametric rate of the estimator, and a bound on the modulus of continuity of the empirical process over the sieve. Lemmas C.2 and C.3 supply the first input from the primitives of the model, and Lemma C.4 supplies the second.

Lemma C.3 (sieve least favorable direction)**.** *For $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ with $|||V||| = 1$, let $a_N(v) \in \mathbb{R}^{d_{K_N}}$ collect the coefficients $Q_{K_N}\widetilde{\delta G}_v$ of the quasi-interpolant of Lemma B.4 applied to an extension $\widetilde{\delta G}_v$ of δG_v to the knot box.*[11] *Under Assumption 5(b), uniformly over such v:*

(i) $\|\nabla(b'_{K_N} a_N(v)) - h_v^{\mathrm{lf}}\|_{L^\infty(\mathcal{Z})} \leq C K_N^{-(1+\alpha)}$.

(ii) $\|\nabla(b'_{K_N} a_N(v))\|_{L^\infty} \leq C$ *and* $\|D^2(b'_{K_N} a_N(v))\|_{L^\infty} \leq C$.

Proof. The extension satisfies $\|\widetilde{\delta G}_v\|_{C^{2,\alpha}(\bar{\mathcal{Z}})} \leq C\|\delta G_v\|_{C^{2,\alpha}(\mathcal{Z})}$, bounded uniformly in v by Lemma C.2(iii). The quasi-interpolant Q_{K_N} is bounded on L^∞, local, and reproduces polynomials of degree at most $q - 1 \geq 2$ (De Boor, 1978, Ch. XII), so on each cell $\widetilde{\delta G}_v - Q_{K_N}\widetilde{\delta G}_v = (\widetilde{\delta G}_v - T) - Q_{K_N}(\widetilde{\delta G}_v - T)$ with T the second-order Taylor polynomial of $\widetilde{\delta G}_v$ at the cell center, and $\|D^i(\widetilde{\delta G}_v - T)\|_{L^\infty} \leq C h^{2+\alpha-i} \|\widetilde{\delta G}_v\|_{C^{2,\alpha}}$ on the cells that determine Q_{K_N} there. The Markov inequality for splines (DeVore and Lorentz, 1993, Ch. 5, Theorem 1.2) bounds $\|D^i Q_{K_N}(\widetilde{\delta G}_v - T)\|_{L^\infty}$ by $C h^{-i} \|Q_{K_N}(\widetilde{\delta G}_v - T)\|_{L^\infty}$. Hence $\|D^i(\widetilde{\delta G}_v - b'_{K_N} a_N(v))\|_{L^\infty} \leq C K_N^{-(2+\alpha-i)}$ for $0 \leq i \leq 2$. Claim (i) is the case $i = 1$ together with $h_v^{\mathrm{lf}} = \nabla \delta G_v$. In particular $\mathrm{dist}_{L^2}(h_v^{\mathrm{lf}}, \mathrm{GradSpl}(K_N)) \leq C K_N^{-(1+\alpha)}$. Claim (ii) follows from the cases $i = 1, 2$ and the uniform $C^{2,\alpha}$ bound on $\widetilde{\delta G}_v$. □

Lemma C.4 (maximal inequality for sieve-indexed classes)**.** *Let $\mathcal{F}_N$ be a class of functions of $\omega = (T, W)$ indexed by a subset of $\mathcal{N} \times \Xi(K_N) \times \Xi(K_N) \times [0, 1]$, with constant envelope F_N, $\sup_{f \in \mathcal{F}_N} \mathrm{Var}(f) \leq \sigma_N^2$ for some $\sigma_N \geq F_N N^{-C}$, and uniform entropy bound $\sup_Q \log N(\epsilon \|F_N\|_{L^2(Q)}, \mathcal{F}_N, L^2(Q)) \leq C d_{K_N} \log(A_N/\epsilon)$ for constants C that do not depend*

[11] Since $\partial\mathcal{Z}$ is of class $C^{2,\alpha}$, δG_v extends to a function $\widetilde{\delta G}_v \in C^{2,\alpha}(\bar{\mathcal{Z}})$ with $\|\widetilde{\delta G}_v\|_{C^{2,\alpha}(\bar{\mathcal{Z}})} \leq C\|\delta G_v\|_{C^{2,\alpha}(\mathcal{Z})}$ (Gilbarg and Trudinger, 2001, Lemma 6.37), and $\widetilde{\delta G}_v$ denotes such an extension. Recall that $\bar{\mathcal{Z}}$ denotes the knot box containing $\mathcal{Z}$, not the closure of $\mathcal{Z}$.

on N and $A_N \le N^C$. Then

$$\mathbb{E}\sup_{\mathcal{F}_N}|(\mathbb{P}_N-\mathbb{P})f| \lesssim \sigma_N K_N\sqrt{\frac{\log N}{N}}+F_N\frac{K_N^2\log N}{N}.$$

Moreover, every class used below is built from finitely many terms of the form $f(\omega)=\sum_t T^t\upsilon(P^t_{\xi_1}(\boldsymbol{\gamma}'W),P^t_{\xi_2}(\boldsymbol{\gamma}'W))\varphi^t(\boldsymbol{\gamma}'W)\psi(W)$ with υ Lipschitz on $[\kappa,1-\kappa]\times[\kappa,1-\kappa]$, ψ bounded, and φ^t lying in a fixed $O(d_{K_N})$-dimensional linear space with bounded envelope and Lipschitz constant in z bounded by a polynomial of K_N. Any such class satisfies the entropy bound.

Proof. The moment bound is Chernozhukov et al. (2014, Theorem 5.2) with variance parameter σ_N, envelope F_N, and entropy dimension $d_{K_N}\lesssim K_N^2$. For the entropy: for fixed $\boldsymbol{\gamma}$, the map $(\xi_1,\xi_2)\mapsto f$ factors through the finite-dimensional space containing $(\varphi^t,P^t_{\xi_1},P^t_{\xi_2})$, and products and Lipschitz transforms of VC-subgraph classes with bounded envelopes retain uniform entropy of order $d_{K_N}\log(1/\epsilon)$ up to constants (Van der Vaart and Wellner, 1996, Lemma 2.6.15, Theorem 2.6.7, and the permanence properties of Section 2.10). Covering the fixed-dimensional $\mathcal{N}$ and the path parameter costs $O(\log(A_N/\epsilon))$, the Lipschitz modulus being polynomial in K_N. □

Lemma C.5 (uniform rates on the core)**.** *Under Assumption 5(a), for every $(\boldsymbol{\gamma},P)\in\mathcal{U}_N$:*

(i) $\|P-P_o\|_{L^\infty(\Omega_{\mathcal{A}})}\lesssim K_N\rho_N$ *and* $\|DP-DP_o\|_{L^\infty(\Omega_{\mathcal{A}})}\lesssim K_N^2\rho_N$*, both $o(1)$.*

(ii) $\mathbb{E}\,\|DP(\boldsymbol{\gamma}'W)-DP_o(\boldsymbol{\gamma}_o'W)\|\lesssim K_N\rho_N+h_N$*, where $h_N\asymp K_N^{-1}$ is the mesh width, the expectation is over W at fixed $(\boldsymbol{\gamma},P)$, and the bound holds uniformly over $\mathcal{U}_N$, a set that shrinks with N.*

Proof. (i): Inside $\mathcal{U}_N$, $\|P-P_o\|_{L^2(\Omega_{\mathcal{A}},dz)}\lesssim\rho_N$ by the definition of L, the index-shift control $2MR_W|||\boldsymbol{\gamma}-\boldsymbol{\gamma}_o|||$, and the density floor on the core. Interposing the approximant P_{ξ° of Lemma B.4 and applying the inverse inequality $\|s\|_{L^\infty}\le DK_N\|s\|_{L^2}$ to the spline $P-P_{\xi^\circ}$ on core-anchored cells gives the first claim. The approximation error of the derivative is $\|DP_o-DP_{\xi^\circ}\|_{L^\infty}\lesssim K_N^{-(k_o-1)}$, and the remaining term $D(P-P_{\xi^\circ})$ costs one further inverse inequality $\|\partial s\|_{L^2}\le DK_N\|s\|_{L^2}$ on top of the first, hence $K_N^2\rho_N$, which dominates $K_N^{-(k_o-1)}$ because $\rho_N\ge K_N^{-k_o}$. Both bounds vanish since $K_N^2\rho_N=\sqrt{K_N^6\log N/N}+K_N^{2-k_o}\to 0$ under Assumption 5(a).

(ii): Split the index shift from the function difference: $\|DP(\boldsymbol{\gamma}'W)-DP(\boldsymbol{\gamma}_o'W)\|\le\|D^2P\|_\infty R_W|||\boldsymbol{\gamma}-\boldsymbol{\gamma}_o|||\lesssim K_N\rho_N$ by Lemma B.3(ii). For the function difference at $\boldsymbol{\gamma}_o'W$,

mediate through P_{ξ°: on $\Omega_{\mathcal{A}}$, the inverse inequality gives $\|D(P - P_{\xi^\circ})\|_{L^2(\Omega_{\mathcal{A}})} \lesssim K_N \rho_N$, and $\|D(P_{\xi^\circ} - P_o)\|_\infty \lesssim K_N^{1-k_o}$. On $\Omega_{\mathcal{W}}$, the integrand is bounded by $4M$ (Lemma B.3(i)) and $\mathbb{P}_{\boldsymbol{\gamma}_o}(\Omega_{\mathcal{W}}) = O(h_N)$. Collecting terms and using $L^1 \le L^2$ on the core gives (ii). □

Lemma C.6 (interiority on interior compacts)**.** *With probability tending to one, $P^t_{\hat{\xi}} \in (\kappa, 1-\kappa)$ on $\Omega_{\mathcal{A}}$, and every constraint of $\Xi(K_N)$ whose defining functional is anchored on $\Omega_{\mathcal{A}}$ is slack at $\hat{\xi}$. The cells have diameter of order $h_N \to 0$, so any compact set in the interior of $\mathcal{Z}$ is contained in $\Omega_{\mathcal{A}}$ for all large N, and the conclusion applies on any such set.*

Proof. By Assumption 4(b) and compactness, $P^t_o \in [\kappa+\eta_0, 1-\kappa-\eta_0]$ on $\bar{\mathcal{Z}}$ for some $\eta_0 > 0$, and the Jacobian constraints likewise hold with fixed margins, by the strict inequalities of Lemma I.3 and the continuity of DP_o on the compact $\bar{\mathcal{Z}}$. Lemma C.5(i) at $\hat{\theta} \in \mathcal{U}_N$ gives $\|P_{\hat{\xi}} - P_o\|_{L^\infty(\Omega_{\mathcal{A}})} = O_{\mathrm{P}}(K_N \rho_N)$ for the overlap constraints and $\|DP_{\hat{\xi}} - DP_o\|_{L^\infty(\Omega_{\mathcal{A}})} = O_{\mathrm{P}}(K_N^2 \rho_N)$ for the Jacobian constraints. Both are $o_{\mathrm{P}}(1)$, hence eventually below every margin, and no constraint involving core values binds. □

Lemma C.7 (strictly feasible anchor)**.** *Let $\xi^\circ = \xi^\circ_{K_N}$ be the approximant of Lemma B.4. There are $\bar{\varepsilon}_0 > 0$ and N_0 such that, for all $N \ge N_0$, all $\bar{\varepsilon} \in (0, \bar{\varepsilon}_0]$, all unit vectors $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, and both signs, $\xi^\circ \pm \bar{\varepsilon} a_N(v) \in \Xi(K_N)$, and $\mathsf{L}((\boldsymbol{\gamma}_o, P_{\xi^\circ \pm \bar{\varepsilon} a_N(v)}), \theta_o) \le C(K_N^{-k_o} + \bar{\varepsilon})$.*

Proof. Every constraint of $\Xi(K_N)$ is a level or first-derivative functional of the propensity, and P_o satisfies the corresponding inequalities with a uniform margin η_0, as in the proofs of Lemmas B.4 and C.6. By Lemma B.4, the same functionals at P_{ξ° are within $CK_N^{-(k_o-1)}$ of their values at P_o. The shift $\pm\bar{\varepsilon} a_N(v)$ moves the propensity by $\pm\bar{\varepsilon}\nabla(b'_{K_N} a_N(v))$ and its Jacobian by $\pm\bar{\varepsilon} D^2(b'_{K_N} a_N(v))$, both bounded by $C\bar{\varepsilon}$ uniformly in v by Lemma C.3(ii). Choosing $\bar{\varepsilon}_0 := \eta_0/(4C)$ and N_0 so that $CK_N^{-(k_o-1)} \le \eta_0/4$ keeps every functional within margin $\eta_0/2$. The L-bound follows from $\|P_{\xi^\circ} - P_o\|_\infty \lesssim K_N^{-k_o}$ and $\|\bar{\varepsilon}\nabla(b'_{K_N} a_N(v))\|_\infty \le C\bar{\varepsilon}$. □

Lemma C.8 (exact identity for the population score)**.** *Let $\theta = (\boldsymbol{\gamma}, P)$ with $P^t(\boldsymbol{\gamma}'\cdot) \in [\kappa, 1-\kappa]$ on $\mathrm{supp}(\boldsymbol{\gamma}'W)$, and let u be a bounded measurable triple of fields on $\mathrm{supp}(W)$ with $\sum_t u^t = 0$. Then*

$$\mathbb{E}\sum_t \frac{T^t}{P^t(\boldsymbol{\gamma}'W)} u^t(W) = -\mathbb{E}\sum_t \frac{P^t(\boldsymbol{\gamma}'W) - P^t_o(\boldsymbol{\gamma}'_o W)}{P^t(\boldsymbol{\gamma}'W)} u^t(W),$$

and consequently

$$\left| \mathbb{E} \sum_t \frac{T^t}{P^t(\boldsymbol{\gamma}'W)} u^t(W) + \langle P(\boldsymbol{\gamma}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), u \rangle_{P_o} \right| \leq 3\kappa^{-2} \|u\|_\infty \mathsf{L}(\theta, \theta_o)^2.$$

Proof. Conditioning on W gives $\mathbb{E}[T^t \,|\, W] = P_o^t(\boldsymbol{\gamma}_o'W)$, and the first display follows from $\sum_t u^t = 0$. For the second, $|1/P^t(\boldsymbol{\gamma}'W) - 1/P_o^t(\boldsymbol{\gamma}_o'W)| \leq \kappa^{-2}|P^t(\boldsymbol{\gamma}'W) - P_o^t(\boldsymbol{\gamma}_o'W)|$, so replacing the weight $1/P^t$ by $1/P_o^t$ in the first display costs at most $\kappa^{-2}\|u\|_\infty \mathbb{E}\sum_t (P^t(\boldsymbol{\gamma}'W) - P_o^t(\boldsymbol{\gamma}_o'W))^2$. □

The expansion compares the criterion at $\hat{\theta}$ with its values along feasible paths toward shifted anchors, in the manner of the sieve likelihood expansions of Shen (1997): his perturbation of the estimate by a small step along the sieve least favorable direction is Lemma C.12, with the anchor and the convex combination replacing his additive perturbation so that the paths stay inside $\Xi(K_N)$; his use of the empirical process modulus is Lemma C.9; and his passage from the expansion to normality and efficiency is Lemma C.13 together with the proof of the theorem. Fix a unit vector $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ with matrix form V, and let $\bar{\varepsilon} := \bar{\varepsilon}_0$ from Lemma C.7. For $\sigma \in \{+1, -1\}$ and $\varepsilon \in [0, N^{-1/2}(\log N)^{-1}]$, define

$$\boldsymbol{\gamma}_\varepsilon^\sigma := \frac{\hat{\boldsymbol{\gamma}} + \sigma\varepsilon V}{|||\hat{\boldsymbol{\gamma}} + \sigma\varepsilon V|||}, \quad \xi_\varepsilon^\sigma := (1 - \varepsilon/\bar{\varepsilon})\,\hat{\xi} + (\varepsilon/\bar{\varepsilon})\,(\xi^\circ - \sigma\bar{\varepsilon} a_N(v))\,, \quad \theta_\varepsilon^\sigma := (\boldsymbol{\gamma}_\varepsilon^\sigma, P_{\xi_\varepsilon^\sigma}).$$

Since $V \in L$, the location normalizations defining Γ are preserved; $\xi_\varepsilon^\sigma \in \Xi(K_N)$ as a convex combination of $\hat{\xi}$ and the feasible point of Lemma C.7; and $\theta_\varepsilon^\sigma \in \mathcal{U}_N$ with probability tending to one for all $\varepsilon \leq N^{-1/2}(\log N)^{-1}$, since the fitted values move by at most $C\varepsilon$. The path derivative of the fitted propensity at $\varepsilon = 0$ is the field $\dot{e}_0^\sigma(w) = \sigma\hat{d}_v(w) + \widehat{\Delta P}^\circ(w)$ where

$$\begin{aligned} \hat{d}_v(w) &:= DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'w)\hat{V}'w - \nabla(b_{K_N}' a_N(v))(\hat{\boldsymbol{\gamma}}'w), \\ \widehat{\Delta P}^\circ(w) &:= \frac{1}{\bar{\varepsilon}}(P_{\xi^\circ} - P_{\hat{\xi}})(\hat{\boldsymbol{\gamma}}'w), \\ \hat{V} &:= V - \langle V, \hat{\boldsymbol{\gamma}} \rangle \hat{\boldsymbol{\gamma}}, \end{aligned}$$

with $\langle A, B \rangle = \mathrm{vec}(A)'\mathrm{vec}(B)$ and $|||\hat{V} - V||| \leq C\rho_N$ on the event $\hat{\boldsymbol{\gamma}} \in \mathcal{N}$. The reference

linear functional and the (sieve) deterministic direction field are

$$\ell[u](\omega) := \sum_t \frac{T^t}{P_o^t(Z)} u^t(W),$$
$$\phi_{N,v}^{\text{eff}}(w) := DP_o(\boldsymbol{\gamma}_o' w)V'w - \nabla(b_{K_N}' a_N(v))(\boldsymbol{\gamma}_o' w),$$

and the population counterpart of $\phi_{N,v}^{\text{eff}}$ is the efficient image ϕ_v^{eff}, with $\|\phi_{N,v}^{\text{eff}} - \phi_v^{\text{eff}}\|_{P_o} \le CK_N^{-(1+\alpha)}$ by Lemma C.3(i).

Lemma C.9 (moduli of the empirical increments)**.** *Uniformly over unit vectors $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, $\sigma \in \{+1, -1\}$, and $\varepsilon \in (0, N^{-1/2}(\log N)^{-1}]$, on the event $\hat{\theta} \in \mathcal{U}_N$:*

(i) $\left|(\mathbb{P}_N - \mathbb{P})[l(\cdot, \theta_\varepsilon^\sigma) - l(\cdot, \hat{\theta})] - \varepsilon(\mathbb{P}_N - \mathbb{P})\ell[\dot{e}_0^\sigma]\right| = \varepsilon \cdot o_{\mathrm{P}}(N^{-1/2}).$

(ii) $\left|(\mathbb{P}_N - \mathbb{P})\ell[\widehat{\Delta P}^\circ]\right| = O_{\mathrm{P}}(\rho_N K_N \sqrt{\log N/N} + K_N^2 \log N/N) = o_{\mathrm{P}}(N^{-1/2}).$

(iii) $\left|(\mathbb{P}_N - \mathbb{P})\ell[\hat{d}_v - \phi_{N,v}^{\text{eff}} - \mathbb{1}_{\mathcal{B}_N} \cdot (DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - DP_o(\boldsymbol{\gamma}_o'\cdot))V'\cdot]\right| = o_{\mathrm{P}}(N^{-1/2})$, *where*

$$\mathcal{B}_N := \{w : \operatorname{dist}(\boldsymbol{\gamma}_o' w, \partial\mathcal{Z}) \le C_* h_N\}$$

and C_* is large enough that $\{\boldsymbol{\gamma}' w \in \Omega_{\mathcal{W}}\} \subseteq \mathcal{B}_N$ for every $\boldsymbol{\gamma} \in \mathcal{N}$.

Proof. Each bound applies Lemma C.4 to a class of the stated structure, indexed by $(\hat{\boldsymbol{\gamma}}, \hat{\xi}, v, \sigma, \varepsilon)$ over $\mathcal{N} \times (\Xi(K_N) \cap \mathcal{U}_N) \times \{|||V||| = 1\} \times \{\pm 1\} \times (0, N^{-1/2}(\log N)^{-1}]$. The entropy requirement holds by the permanence argument there, so only envelopes and variances are computed.

Step 1 (Claim (ii)): The class member is $\ell[\widehat{\Delta P}^\circ]$ with $\|\widehat{\Delta P}^\circ\|_\infty \le 2/\bar{\varepsilon}$ and $\operatorname{Var}(\ell[\widehat{\Delta P}^\circ]) \le \kappa^{-2}\mathbb{E}\|(P_{\xi^\circ} - P_{\hat{\xi}})(\hat{\boldsymbol{\gamma}}'W)\|^2/\bar{\varepsilon}^2 \le C\rho_N^2$, since L-distances to θ_o of both $(\hat{\boldsymbol{\gamma}}, P_{\xi^\circ})$ and $\hat{\theta}$ are $O(\rho_N)$ on $\mathcal{U}_N$ by Lemma C.7. Lemma C.4 gives the display, and $\sqrt{N}(\rho_N K_N \sqrt{\log N/N} + K_N^2 \log N/N) = K_N^2 \log N/\sqrt{N} \cdot O(1) \to 0$ under Assumption 5(a).

Step 2 (Claim (i)): For $t \in \{0, 1, 2\}$, let $\Delta_\varepsilon^t(w) := P_{\xi_\varepsilon^\sigma}^t((\boldsymbol{\gamma}_\varepsilon^\sigma)'w) - P_{\hat{\xi}}^t(\hat{\boldsymbol{\gamma}}'w)$ be the change in the fitted propensity of treatment t from $\hat{\theta}$ to $\theta_\varepsilon^\sigma$, and let $\Delta_\varepsilon := (\Delta_\varepsilon^0, \Delta_\varepsilon^1, \Delta_\varepsilon^2)$. Then $\|\Delta_\varepsilon\|_\infty \le C\varepsilon$ and $\|\Delta_\varepsilon - \varepsilon\dot{e}_0^\sigma\|_\infty \le C\varepsilon^2 K_N$, the second bound because the path curvature

of the fitted values is bounded by $\|D^2 P\|_\infty$ and Lemma B.3(ii). Decompose

$$l(\cdot,\theta_\varepsilon^\sigma) - l(\cdot,\hat\theta) - \varepsilon\ell[\dot e_0^\sigma] = \underbrace{\left[l(\cdot,\theta_\varepsilon^\sigma) - l(\cdot,\hat\theta) - \sum_t \frac{T^t\Delta_\varepsilon^t}{P_{\hat\xi}^t(\hat{\boldsymbol\gamma}'W)}\right]}_{g_1} + \underbrace{\sum_t \frac{T^t(\Delta_\varepsilon^t - \varepsilon\dot e_0^{\sigma,t})}{P_{\hat\xi}^t(\hat{\boldsymbol\gamma}'W)}}_{g_2} + \underbrace{\varepsilon\sum_t T^t\dot e_0^{\sigma,t}\left(\frac{1}{P_{\hat\xi}^t(\hat{\boldsymbol\gamma}'W)} - \frac{1}{P_o^t(Z)}\right)}_{g_3}.$$

A second-order Taylor bound for the logarithm on $[\kappa, 1-\kappa]$ gives $\|g_1\|_\infty \le \kappa^{-2}\|\Delta_\varepsilon\|_\infty^2 \le C\varepsilon^2$. Also $\|g_2\|_\infty \le C\varepsilon^2 K_N$ and $|g_3| \le C\varepsilon\kappa^{-2}\max_t|P_{\hat\xi}^t(\hat{\boldsymbol\gamma}'W) - P_o^t(Z)|$, so $\mathrm{Var}(g_3) \le C\varepsilon^2\rho_N^2$. Lemma C.4 applied to the three classes yields

$$\varepsilon\left[\varepsilon K_N^2\sqrt{\log N/N} + \rho_N K_N\sqrt{\log N/N} + K_N^2\log N/N\right]$$

up to constants, and each bracket term is $o_{\mathrm{P}}(N^{-1/2})$: the first since $\varepsilon K_N^2\sqrt{\log N/N} \le N^{-1/2}K_N^2/\sqrt{N\log N}$ and $K_N^2/\sqrt{N\log N} \to 0$, the others as in Step 1.

Step 3 (Claim (iii)): The class member has three parts,

$$\nabla(b'_{K_N}a_N(v))(\boldsymbol\gamma_o'w) - \nabla(b'_{K_N}a_N(v))(\hat{\boldsymbol\gamma}'w),$$

$DP_{\hat\xi}(\hat{\boldsymbol\gamma}'w)(\hat V - V)'w$, and $\mathbb{1}_{\mathcal{B}_N^c}(DP_{\hat\xi}(\hat{\boldsymbol\gamma}'w) - DP_o(\boldsymbol\gamma_o'w))V'w$. The first two are bounded by $C\rho_N$ pointwise, by Lemma C.3(ii) and $\|\!|\hat V - V|\!\| \le C\rho_N$. For the third, $\mathcal{B}_N^c$ forces $\hat{\boldsymbol\gamma}'w$ and $\boldsymbol\gamma_o'w$ into $\Omega_{\mathcal{A}}$ up to the enlargement absorbed in C_*, so the integrand is bounded by the core rate: pointwise by $K_N^2\rho_N$ and in $L^2(\mathbb{P})$ by $CK_N\rho_N$, by Lemma C.5. Lemma C.4 with $\sigma_N = CK_N\rho_N$ and $F_N = CK_N^2\rho_N$ gives $O_{\mathrm{P}}(K_N^2\rho_N\sqrt{\log N/N} + K_N^4\rho_N\log N/N)$, and both terms times $\sqrt{N}$ vanish under Assumption 5(a). □

Lemma C.10 (cells meeting the boundary)**.** *Let $\partial\mathcal{Z}$ be of class C^2. There are constants C_1, C_2, C_3, $c_4 > 0$ and K_0, depending only on $\partial\mathcal{Z}$ and the mesh constants c_1, c_2 of Assumption 4(c), such that for $K_N \ge K_0$:*

(i) Every cell Q with $\mathfrak{f}_Q \ge C_1K_N^{-1}$ contains a rectangle $R_Q \subseteq Q\cap\mathcal{Z}$ with sides parallel to the axes and $|R_Q| \ge |Q\cap\mathcal{Z}|/4$.

(ii) Every cell Q with $0 < \mathfrak{f}_Q < 1/8$ shares a side with a cell Q' with $\mathfrak{f}_{Q'} \ge c_4$, and every point of $Q\cap\mathcal{Z}$ lies within distance $C_2(\mathfrak{f}_Q^{1/2} + K_N^{-1/2})K_N^{-1}$ of that shared side $Q\cap Q'$.

(iii) $|\Omega_{\mathcal{W}}| \le C_3 h_N$, *and the cells with* $0 < \mathfrak{f}_Q < 1$ *number at most* $C_3 K_N$.

Proof. Write $h_N := K_N^{-1}$, so that cells have sides in $[c_1 h_N, c_2 h_N]$ and $\operatorname{diam}(Q) \le 2c_2 h_N$.

Step 1 (Local graph representation): This step constructs the coordinates in which $\partial\mathcal{Z}$ is a graph and $\mathcal{Z}$ lies above it, with constants that do not depend on the base point. Throughout, $\varkappa = \varkappa(z)$ denotes the curvature of $\partial\mathcal{Z}$ at z: a component of $\partial\mathcal{Z}$ parametrized by arclength as $\tau \mapsto z(\tau)$ has unit tangent $z'(\tau)$, and $\varkappa(\tau) := |z''(\tau)|$. Assumption 5(b) makes $\partial\mathcal{Z}$ a compact one-dimensional manifold of class C^2, hence a finite disjoint union of simple closed curves of class C^2, on which $\varkappa$ is continuous. Set $\varkappa_{\max} := \max_{z\in\partial\mathcal{Z}} \varkappa(z)$, which is finite by compactness and positive: were $\varkappa$ identically zero, each component would satisfy $z'' \equiv 0$ and hence $z(\tau) = z(0) + \tau z'(0)$, which is not periodic, while the arclength parametrization of a closed curve is. Write $\tau_0 := 1/(2\varkappa_{\max})$ for the window in the arclength parameter used below.

Coordinates. Fix $z_0 \in \partial\mathcal{Z}$ and let $z(\cdot)$ parametrize its component by arclength with $z(0) = z_0$. Let $\mathsf{e}_\mathsf{s} := z'(0)$ and let e_y be a unit vector orthogonal to it, the choice of sign being fixed at the end of the step. For a point z, write (s, y) for the coordinates of $z - z_0$ in the orthonormal basis $(\mathsf{e}_\mathsf{s}, \mathsf{e}_\mathsf{y})$, and along the curve write $\mathsf{s}(\tau) := \langle z(\tau) - z_0, \mathsf{e}_\mathsf{s}\rangle$ and $\mathsf{y}(\tau) := \langle z(\tau) - z_0, \mathsf{e}_\mathsf{y}\rangle$.

The arc through z_0 *is a graph.* Since $|z'(\tau) - z'(0)| \le \int_0^{|\tau|} \varkappa \le \varkappa_{\max}|\tau|$, for $|\tau| \le \tau_0$,

$$\mathsf{s}'(\tau) = \langle z'(\tau), \mathsf{e}_\mathsf{s}\rangle \ge 1 - \varkappa_{\max}|\tau| \ge \tfrac{1}{2}, \qquad |\mathsf{y}'(\tau)| = |\langle z'(\tau) - z'(0), \mathsf{e}_\mathsf{y}\rangle| \le \varkappa_{\max}|\tau| \le \tfrac{1}{2},$$

using $\langle z'(0), \mathsf{e}_\mathsf{s}\rangle = 1$ and $\langle z'(0), \mathsf{e}_\mathsf{y}\rangle = 0$. So s is strictly increasing on $[-\tau_0, \tau_0]$ with $\mathsf{s}(\pm\tau_0) \gtrless \pm\tau_0/2$, and it has a C^2 inverse $\tau(\cdot)$ there by the inverse function theorem. Setting $u(\mathsf{s}) := \mathsf{y}(\tau(\mathsf{s}))$ writes the arc $z([-\tau_0, \tau_0])$ as the graph $\{\mathsf{y} = u(\mathsf{s})\}$ of a C^2 function over an interval of s that contains $[-\tau_0/2, \tau_0/2]$.

Properties of u. $u(0) = \mathsf{y}(0) = 0$ because the coordinates are centred at $z_0 = z(0)$, and $u'(0) = \mathsf{y}'(0)/\mathsf{s}'(0) = \langle z'(0), \mathsf{e}_\mathsf{y}\rangle = 0$ because the first coordinate axis is the tangent direction at z_0, which is orthogonal to e_y. Neither is a restriction on $\partial\mathcal{Z}$. Both stem from the choice of coordinates. On $[-\tau_0, \tau_0]$, $|u'| = |\mathsf{y}'|/\mathsf{s}' \le 1$ by the two bounds above, and the curvature of a graph is $\varkappa = |u''|(1 + (u')^2)^{-3/2}$, so $|u''| \le 2^{3/2}\varkappa_{\max} \le \bar{\varkappa} := 3\varkappa_{\max}$. Integrating $|u''| \le \bar{\varkappa}$ twice from $u(0) = u'(0) = 0$ gives $|u(\mathsf{s})| \le \bar{\varkappa}\mathsf{s}^2/2$.

No other part of the boundary is close to z_0. The graph just constructed represents one arc, and a second arc of $\partial\mathcal{Z}$, far from z_0 along the curve, could still pass close to z_0 in the plane. The following excludes that possibility, uniformly in z_0. Let ϱ be the minimum

of $|z - z'|$ over the pairs $(z, z') \in \partial\mathcal{Z} \times \partial\mathcal{Z}$ that lie either on different components or on the same component at arclength distance at least τ_0. Those pairs form a closed subset of the compact $\partial\mathcal{Z} \times \partial\mathcal{Z}$, so the minimum is attained, and it is positive because distinct components are disjoint and each component is a simple curve, so no such pair has $z = z'$. Hence any $z_1 \in \partial\mathcal{Z}$ with $|z_1 - z_0| < \varrho$ lies on the component of z_0 at arclength distance below τ_0, that is, on the arc $z([-\tau_0, \tau_0])$.

The neighbourhood. Set $r_0 := \min\{\varrho/2,\ \tau_0/2,\ 1/(2\bar{\varkappa})\} > 0$ and let Ω_0 be the (s, y)-coordinate square $\{|\mathsf{s}| < r_0\} \cap \{|\mathsf{y}| < r_0\}$, which trivially contains $B(z_0, r_0)$ and is contained in $B(z_0, \sqrt{2}r_0)$. Every $z \in \Omega_0 \cap \partial\mathcal{Z}$ has $|z - z_0| < \sqrt{2}r_0 \le \varrho/\sqrt{2} < \varrho$, hence lies on the arc, and its first coordinate satisfies $|\mathsf{s}| < r_0 \le \tau_0/2$, so it lies on the graph. Conversely, every graph point with $|\mathsf{s}| < r_0$ lies on the arc, and $|u(\mathsf{s})| \le \bar{\varkappa} r_0^2/2 \le r_0/4$ places it in Ω_0. Therefore $\partial\mathcal{Z} \cap \Omega_0 = \{\mathsf{y} = u(\mathsf{s})\} \cap \Omega_0$.

Which side is $\mathcal{Z}$. Write $\Omega_0^+ := \{\mathsf{y} > u(\mathsf{s})\} \cap \Omega_0$ and $\Omega_0^- := \{\mathsf{y} < u(\mathsf{s})\} \cap \Omega_0$. Each is path connected: since $|u| \le r_0/4$ on $|\mathsf{s}| < r_0$, any point of Ω_0^+ is joined inside Ω_0^+ to $(0, 3r_0/4)$ by a vertical segment followed by a horizontal one at height $3r_0/4$, and likewise below. Both are disjoint from $\partial\mathcal{Z}$, so each lies either in $\operatorname{Int}\mathcal{Z}$ or in the complement of $\mathcal{Z}$, these two sets being open, disjoint, and covering the complement of $\partial\mathcal{Z}$. They cannot lie on the same side: $z_0 \in \partial\mathcal{Z}$ puts points of the complement of $\mathcal{Z}$ in every neighbourhood of z_0, and $\mathcal{Z}$ is the closure of its interior by Assumption 4(d), which puts points of $\operatorname{Int}\mathcal{Z}$ there as well. Fixing the sign of e_y so that $\Omega_0^+ \subseteq \operatorname{Int}\mathcal{Z}$ therefore gives $\mathcal{Z} \cap \Omega_0 = \{\mathsf{y} \ge u(\mathsf{s})\} \cap \Omega_0$, and e_y is the inward normal at z_0.

The sandwich. Let K_0 be chosen to satisfy $4c_2h_N \le r_0/2$ for $K_N \ge K_0$, so that a cell containing z_0, and each of its mesh neighbours, lies within distance $4c_2h_N$ of z_0. Both coordinates of such a point are then below $r_0/2$ in modulus and the cell must lie inside Ω_0. Fix a cell Q with $0 < \mathfrak{f}_Q < 1$, so that Q meets $\partial\mathcal{Z}$. Pick $z_0 \in Q \cap \partial\mathcal{Z}$, let $H := \{\mathsf{y} \ge 0\}$ be the closed half-plane bounded by the tangent line at z_0 on the side of $\mathcal{Z}$, and let $S := \{|\mathsf{y}| \le 2\bar{\varkappa}c_2^2h_N^2\}$, a strip that contains the graph over $|\mathsf{s}| \le 2c_2h_N$ by $|u(\mathsf{s})| \le \bar{\varkappa}\mathsf{s}^2/2$. Then, for every rectangle $Q^* \subseteq \Omega_0$ with $\operatorname{diam}(Q^*) \le 2c_2h_N$ that contains z_0 or is adjacent to Q,

$$\begin{aligned} &(H \setminus S) \cap Q^* \subseteq \mathcal{Z} \cap Q^* \subseteq (H \cup S) \cap Q^*, \\ &|S \cap Q^*| \le 8\bar{\varkappa}c_2^3h_N^3 \le C_0h_N|Q^*|, \end{aligned} \tag{C.2}$$

with $C_0 := 8\bar{\varkappa}c_2^3/c_1^2$. In words, the part of Q^* that is above the strip is contained in $\mathcal{Z}$,

and the part of Q^* that is below the strip is contained in the complement of $\mathcal{Z}$. The two inclusions hold because a point of Q^* off the strip is on the same side of the graph as of the tangent line. The bound on $|S \cap Q^*|$ arises because the strip has width $4\bar{\varkappa}c_2^2h_N^2$ and Q^* has diameter at most $2c_2h_N$. In particular $\mathfrak{g}_Q := |H \cap Q|/|Q|$ satisfies $|\mathfrak{f}_Q - \mathfrak{g}_Q| \le C_0h_N$.

Step 2 (Rectangles in a line-cut rectangle): Let $Q^* = [0,a] \times [0,b]$ be a rectangle and H^* a closed half-plane whose boundary line meets Q^*. Then $Q^* \cap H^*$ contains a rectangle with sides parallel to the axes and area at least $|Q^* \cap H^*|/2$, whichever of the two pieces of Q^* the half-plane selects, by the following geometric argument: If the line meets two adjacent sides of Q^*, at distances $a' \le a$ and $b' \le b$ from their common corner, then $Q^* \cap H^*$ is either the triangle with legs a', b', which contains $[0,a'/2] \times [0,b'/2]$ of area $a'b'/4 = |Q^* \cap H^*|/2$, or its complement $\{\mathsf{s}/a' + \mathsf{y}/b' \ge 1\} \cap Q^*$, which contains $[a'/2, a] \times [b'/2, b]$, and $(a - a'/2)(b - b'/2) - (ab - a'b'/2)/2 = (a - a')(b - b')/2 \ge 0$. If the line meets two opposite sides, say the vertical ones at heights $\mathsf{y}_1 \le \mathsf{y}_2$, and $Q^* \cap H^*$ lies below the line, then it contains $[a/2, a] \times [0, (\mathsf{y}_1 + \mathsf{y}_2)/2]$, of area $a(\mathsf{y}_1 + \mathsf{y}_2)/4 = |Q^* \cap H^*|/2$. The case above the line and the case of horizontal sides are the same after a reflection or an exchange of the axes. A line through a corner is a limiting case of one of these.

Step 3 (Claim (i)): If $\mathfrak{f}_Q = 1$, $Q \subseteq \mathcal{Z}$ up to a null set and $R_Q := Q$ serves, so let $C_1h_N \le \mathfrak{f}_Q < 1$, which makes Q meet $\partial\mathcal{Z}$ and puts the construction of Step 1 at its disposal. Let $C_1 := 4C_0$ and let H' be the half-plane H shifted by $2\bar{\varkappa}c_2^2h_N^2$ into its interior, so that $H' \cap Q = (H \setminus S) \cap Q \subseteq \mathcal{Z} \cap Q$ by (C.2). Then $|H' \cap Q| \ge |H \cap Q| - |S \cap Q| \ge |\mathcal{Z} \cap Q| - |S \cap Q| \ge (\mathfrak{f}_Q - 2C_0h_N)|Q| \ge \mathfrak{f}_Q|Q|/2$, and if the boundary line of H' does not meet Q then $H' \cap Q = Q$. Step 2 applied to $H' \cap Q$ gives a rectangle $R_Q \subseteq \mathcal{Z} \cap Q$ with $|R_Q| \ge \mathfrak{f}_Q|Q|/4 = |Q \cap \mathcal{Z}|/4$ whose sides are parallel to the axes, as required.

Step 4 (Claim (ii)): Let $0 < \mathfrak{f}_Q < 1/8$, so that $\mathfrak{g}_Q < 1/8 + C_0h_N \le 1/4$ for $K_N \ge K_0$. Map Q onto the unit square by the affine map $(\mathsf{s}, \mathsf{y}) \mapsto (\mathsf{s}/a, \mathsf{y}/b)$, where $a, b \in [c_1h_N, c_2h_N]$ are the sides of Q. The images of the tangent line ∂H and of H are a line and a half-plane $\tilde{H}$, and $\tilde{H} \cap [0,1]^2$ has area $\mathfrak{g}_Q < 1/4$. Exchanging the axes if necessary, the image line has slope in $[-1,1]$ and is therefore the graph of an affine function $\tilde{u}$, and reflecting $\tilde{\mathsf{y}} \mapsto 1 - \tilde{\mathsf{y}}$ if necessary, $\tilde{H} = \{\tilde{\mathsf{y}} \le \tilde{u}(\tilde{\mathsf{s}})\}$. Write side for the side of Q that corresponds to $\{\tilde{\mathsf{y}} = 0\}$. The height $\mathsf{t}(\tilde{\mathsf{s}}) := \min(\tilde{u}(\tilde{\mathsf{s}}), 1)^+$ of $\tilde{H} \cap [0,1]^2$ over $\tilde{\mathsf{s}} \in [0,1]$ is 1-Lipschitz with maximum $T \in [0,1]$ attained at some $\tilde{\mathsf{s}}^*$, so $\mathfrak{g}_Q = \int_0^1 \mathsf{t} \ge \int_0^1 (T - |\tilde{\mathsf{s}} - \tilde{\mathsf{s}}^*|)^+ d\tilde{\mathsf{s}} \ge T^2/4$, the last inequality because $[0,1]$ contains an interval of length $\min(T, 1/2)$ on one side of $\tilde{\mathsf{s}}^*$. Hence $T \le 2\mathfrak{g}_Q^{1/2} \le 2\mathfrak{f}_Q^{1/2} + 2(C_0h_N)^{1/2}$. By (C.2), $\mathcal{Z} \cap Q \subseteq (H \cup S) \cap Q$, whose points are within distance $(T + 2\bar{\varkappa}c_2^2h_N)\max(a,b)$ of side, and $\max(a,b) \le c_2h_N$ turns this into the bound

of (ii) with C_2 depending on $\bar{\varkappa}, c_1, c_2$. For the neighbour, the region beyond side within Ω_0 meets $H \setminus S$ in a set of positive measure, hence meets $\mathcal{Z}$, so side is not on the boundary of the knot box $\bar{\mathcal{Z}} \supseteq \mathcal{Z}$, and the cell Q' across it exists, with $\mathsf{side} = Q \cap Q'$. In the coordinates above it is $[0,1] \times [-\lambda, 0]$ for some $\lambda \in [c_1/c_2, c_2/c_1]$. Since $\tilde{u}(\tilde{\mathsf{s}}) \geq T - |\tilde{\mathsf{s}} - \tilde{\mathsf{s}}^*| \geq -|\tilde{\mathsf{s}} - \tilde{\mathsf{s}}^*|$, the set $\tilde{H} \cap ([0,1] \times [-\lambda, 0])$ contains $\{-\lambda \leq \tilde{\mathsf{y}} \leq -|\tilde{\mathsf{s}} - \tilde{\mathsf{s}}^*|\}$, of area at least $\min(\lambda, 1)^2/4$ by the computation above. Returning to the original coordinates and using (C.2) for $Q^* = Q'$, $|\mathcal{Z} \cap Q'| \geq (\min(\lambda, 1)^2/4)ab - C_0 h_N |Q'|$, so $\mathfrak{f}_{Q'} \geq \min(\lambda, 1)^2/(4\lambda) - C_0 h_N \geq c_4 := (c_1/c_2)^3/8$ for $K_N \geq K_0$.

Step 5 (Claim (iii)): A cell with $0 < \mathfrak{f}_Q < 1$ meets $\partial\mathcal{Z}$, so each of its points lies within $\operatorname{diam}(Q) \leq 2c_2 h_N$ of $\partial\mathcal{Z}$. A cell of $\Omega_{\mathcal{W}}$ is such a cell or a mesh neighbour of one, so each of its points lies within $r := 4c_2 h_N$, which is at most r_0 for $K_N \geq K_0$. Every point within distance r of $\partial\mathcal{Z}$ has a nearest point on $\partial\mathcal{Z}$, and the segment joining the two meets $\partial\mathcal{Z}$ orthogonally there, so the set of such points is contained in the image of $\partial\mathcal{Z} \times [-r, r]$ under $(z, t) \mapsto z + t\mathbf{n}(z)$. In the arclength parametrization that map has Jacobian $|1 - t\varkappa| \leq 1 + r\bar{\varkappa} \leq 2$ for $K_N \geq K_0$, so the area formula bounds the measure of that set by $4r|\partial\mathcal{Z}|$, with $|\partial\mathcal{Z}|$ the length of the boundary. Hence $|\Omega_{\mathcal{W}}| \leq 16c_2|\partial\mathcal{Z}|h_N$, and since every cell has area at least $c_1^2 h_N^2$ the cells of $\Omega_{\mathcal{W}}$, among them every cell with $0 < \mathfrak{f}_Q < 1$, number at most $16c_2|\partial\mathcal{Z}|c_1^{-2}K_N$. □

Lemma C.11 (boundary layer interaction)**.** *Let*

$$\widehat{\Delta P}(z) := P_{\hat{\xi}}(z) - P_o(z),$$
$$\widehat{\Delta D}(w) := DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}' w) - DP_o(\boldsymbol{\gamma}_o' w).$$

Here $\widehat{\Delta P}$ is a field of $z \in \mathcal{Z}$ and $\widehat{\Delta D}$ a field of w. Under Assumption 5(a) and (b), uniformly over unit vectors $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$:

(i) $\left|\langle \widehat{\Delta P},\ \mathbb{1}_{\mathcal{B}_N} \cdot (\widehat{\Delta D}\, V' \cdot) \rangle_{P_o}\right| = o_{\mathrm{P}}(N^{-1/2})$.

(ii) $\left|(\mathbb{P}_N - \mathbb{P})\ell[\mathbb{1}_{\mathcal{B}_N} \cdot (\widehat{\Delta D}\, V' \cdot)]\right| = o_{\mathrm{P}}(N^{-1/2})$.

Proof. All statements are conditional on the event $\hat{\theta} \in \mathcal{U}_N$, of probability tending to one.

Step 1 (Derivative rate on the boundary layer): On each cell Q, every component of $P_{\hat{\xi}} - P_{\xi^\circ}$ is a polynomial of degree at most $q-1$ in *each* coordinate, and across a common side of two cells these components are of class C^{q-3}, being derivatives of the C^{q-2} surplus spline. Write p for the restriction of a component to Q, p' for its restriction to a neighboring cell Q', and

$\mathfrak{f}_Q$ for the covered fraction of Q. Assumption 5(b) makes $\partial\mathcal{Z}$ of class C^2, so Lemma C.10 applies, with constants C_1, C_2, c_4. Fix a threshold $\mathfrak{f}_* \in (C_1 K_N^{-1}, 1/8)$, specified at the end of the step, and call a cell thick if $\mathfrak{f}_Q \geq \mathfrak{f}_*$ and thin if $0 < \mathfrak{f}_Q < \mathfrak{f}_*$. Three inequalities are used below. Let π be a polynomial in two variables of degree at most $q-1$ in each coordinate, the class just described, and let B be a box with both sides of order h_N. The Markov inequality $\|D\pi\|_{L^2(B)} \leq C(q) h_N^{-1} \|\pi\|_{L^2(B)}$ and the inverse inequality $\sup_B |\pi| \leq C(q)|B|^{-1/2}\|\pi\|_{L^2(B)}$ are the standard compactness argument. On the unit square, $\pi \mapsto \|D\pi\|_{L^2}$ and $\pi \mapsto \sup|\pi|$ are continuous on this class, which has dimension q^2 and carries $\|\cdot\|_{L^2}$ as a norm, so each is bounded there by $C(q)\|\pi\|_{L^2}$, and the affine change of variables onto B supplies the powers of h_N. Reading the degree as the total degree instead gives a subspace of the same class, on which these constants still serve. The third inequality is univariate, the Remez inequality $\sup_J |\pi| \leq T_{q-1}(2|J|/|J'| - 1) \sup_{J'} |\pi| \leq (4|J|/|J'|)^{q-1} \sup_{J'} |\pi|$ for an interval J, a subinterval J', and the Chebyshev polynomial T_{q-1} of degree $q-1$ (Brudnyi and Ganzburg, 1973).

Thick cells. By Lemma C.10(i), $Q \cap \mathcal{Z}$ contains a rectangle R_Q with sides parallel to the axes and $|R_Q| \geq \mathfrak{f}_Q|Q|/4$. Let $a, b \in (0, 1]$ be the ratios of its sides to those of Q, so that $ab \geq \mathfrak{f}_Q/4$. Applying the Remez inequality in each coordinate and then the inverse inequality on R_Q,

$$\begin{aligned}\|p\|_{L^2(Q)} \leq |Q|^{1/2} \sup_Q |p| &\leq |Q|^{1/2} \left(\frac{16}{ab}\right)^{q-1} \sup_{R_Q} |p| \\ &\leq C(q)(ab)^{-(q-1/2)} \|p\|_{L^2(R_Q)} \leq C\mathfrak{f}_*^{-(q-1/2)} \|p\|_{L^2(Q\cap\mathcal{Z})},\end{aligned}$$

and the Markov inequality gives $\|Dp\|_{L^2(Q\cap\mathcal{Z})} \leq C\mathfrak{f}_*^{-(q-1/2)} K_N \|p\|_{L^2(Q\cap\mathcal{Z})}$.

Thin cells. By Lemma C.10(ii), Q shares a side $Q \cap Q'$ with a cell Q' with $\mathfrak{f}_{Q'} \geq c_4$, and $Q \cap \mathcal{Z}$ lies within distance $\delta_Q h$ of that side, where $\delta_Q := C(\mathfrak{f}_Q^{1/2} + K_N^{-1/2})$. Let s be a coordinate along the line through $Q \cap Q'$ and y the coordinate normal to it, oriented so that $\mathsf{y} \geq 0$ on Q, which is available because $Q \cap Q'$ is a side of Q and Q lies on one side of that line. On Q, y is then the distance to the line. Since p and p' agree on $Q \cap Q'$ together with their normal derivatives up to order $q-3$, we may write $p = p_0 + p_1$ with

$$p_0(\mathsf{s}, \mathsf{y}) := \sum_{j=0}^{q-3} \frac{\mathsf{y}^j}{j!} \partial_{\mathsf{y}}^j p'(\mathsf{s}, 0), \qquad p_1(\mathsf{s}, \mathsf{y}) := \sum_{j=q-2}^{q-1} c_j(\mathsf{s}) \mathsf{y}^j,$$

where the c_j are polynomials of degree at most $q-1$. The thick-cell bound applied

to Q', whose fraction c_4 exceeds $\mathfrak{f}_*$ for large N, together with the Markov and inverse inequalities, gives $\|D^k p'\|_{L^\infty(Q')} \le Ch^{-k-1}\|p'\|_{L^2(Q')} \le Ch^{-k-1}c_4^{-(q-1/2)}\|p'\|_{L^2(Q'\cap\mathcal{Z})}$ for $k \le q$, hence $|Dp_0| \le A_Q := Ch^{-2}\|p'\|_{L^2(Q'\cap\mathcal{Z})}$ on Q. On Q, $|Dp| \le 4M$ by Lemma B.3(i), so $|Dp_1| \le 4M + A_Q$ on Q. For fixed s, $\partial_{\mathsf{y}} p_1(\mathsf{s},\cdot)$ is a polynomial in y on an interval of length at least $c_1 h_N$ bounded by $4M + A_Q$, so its coefficients satisfy $j|c_j(\mathsf{s})|h_N^{j-1} \le C(4M + A_Q)$, and the Markov inequality in s gives $|c_j'(\mathsf{s})|h_N^j \le C(4M + A_Q)$. On $Q \cap \mathcal{Z}$, where $\mathsf{y} \le \delta_Q h$, therefore $|\partial_{\mathsf{y}} p_1| \le C(M + A_Q)\delta_Q^{q-3}$ and $|\partial_{\mathsf{s}} p_1| \le C(M + A_Q)\delta_Q^{q-2}$, and

$$\|Dp\|^2_{L^2(Q\cap\mathcal{Z})} \le C|Q \cap \mathcal{Z}|(A_Q^2 + M^2\delta_Q^{2q-6}) \le C\mathfrak{f}_Q K_N^2 \|p'\|^2_{L^2(Q'\cap\mathcal{Z})} + CM^2\mathfrak{f}_Q\delta_Q^{2q-6}h_N^2.$$

Summation. Every cell that meets $\{z : \mathrm{dist}(z, \partial\mathcal{Z}) \le C_* h_N\}$ is thick, thin, or empty, and a cell is the neighbor of at most four thin cells. Summing the two bounds, with $p_Z \in [\underline{p}, \bar{p}]$ on $\mathcal{Z}$ by Assumption 4(f), and using $\delta_Q^{2q-6} \le C(\mathfrak{f}_Q^{q-3} + K_N^{-(q-3)})$,

$$\begin{aligned}\left\|\mathbb{1}_{\mathcal{B}_N}(\cdot)D(P_{\hat\xi} - P_{\xi^\circ})(\boldsymbol{\gamma}_o'\cdot)\right\|^2_{L^2(\mathbb{P})} &\le C\mathfrak{f}_*^{-(2q-1)}K_N^2\left\|(P_{\hat\xi} - P_{\xi^\circ})(\boldsymbol{\gamma}_o'\cdot)\right\|^2_{L^2(\mathbb{P})} \\ &\quad + C\sum_{Q\ \text{thin}}\mathfrak{f}_Q^{q-2}h_N^2 + CK_N^{-(q-3)}h_N^2\sum_{Q\ \text{thin}}\mathfrak{f}_Q.\end{aligned}$$

By Lemma C.10(iii) the cells with $0 < \mathfrak{f}_Q < 1$ number at most $C_3 K_N$, and $\mathfrak{f}_Q < \mathfrak{f}_*$ on a thin cell, so

$$\sum_{Q\ \text{thin}}\mathfrak{f}_Q^{q-2}h_N^2 \le C\mathfrak{f}_*^{q-2}K_N^{-1}, \qquad \sum_{Q\ \text{thin}}\mathfrak{f}_Q \le CK_N\mathfrak{f}_*,$$

so the last term is at most $C\mathfrak{f}_* K_N^{-(q-2)}$. The first term is bounded through the identity $P_{\hat\xi} - P_{\xi^\circ} = \widehat{\Delta P} - (P_{\xi^\circ} - P_o)$ of fields of z, where $\|\widehat{\Delta P}(Z)\|_{L^2(\mathbb{P})} \le \mathsf{L}(\hat\theta, \theta_o) + 2MR_W\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\| \le C\rho_N$ by Lemma B.3(i) and $\|P_{\xi^\circ} - P_o\|_\infty \lesssim K_N^{-k_o}$ by Lemma B.4. On $\mathcal{B}_N$, $\widehat{\Delta D}(w)$ differs from $D(P_{\hat\xi} - P_{\xi^\circ})(\boldsymbol{\gamma}_o' w)$ by the index shift $DP_{\hat\xi}(\hat{\boldsymbol{\gamma}}' w) - DP_{\hat\xi}(\boldsymbol{\gamma}_o' w)$, bounded by $\|D^2 P_{\hat\xi}\|_\infty R_W\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\| \le CK_N\rho_N$ by Lemma B.3(ii) on the set $\mathcal{B}_N$, of probability $O(h_N)$, and $D(P_{\xi^\circ} - P_o)(\boldsymbol{\gamma}_o' w)$ is bounded by $CK_N^{-(k_o-1)}$. Put together,

$$\begin{aligned}\sigma_N := \|\mathbb{1}_{\mathcal{B}_N}\widehat{\Delta D}\|_{L^2(\mathbb{P})} &\le C(\mathfrak{f}_*^{-(q-1/2)}K_N\rho_N + \mathfrak{f}_*^{(q-2)/2}K_N^{-1/2} + \mathfrak{f}_*^{1/2}K_N^{-(q-2)/2}) \\ &\quad + C(K_N^{1/2}\rho_N + K_N^{-(k_o-1)}).\end{aligned}$$

Now take $\mathfrak{f}_*^{q-2} := (K_N(\log N)^2)^{-1}$, which lies in $(C_1 K_N^{-1}, 1/8)$ for large N since $q \ge 5$, and note that $\mathfrak{f}_*^{-(q-1/2)} = (K_N(\log N)^2)^{(q-1/2)/(q-2)} \le (K_N(\log N)^2)^{3/2}$ for $q \ge 5$. Multiplying

the five terms by $K_N\sqrt{\log N}$ and using $\rho_N \le K_N\sqrt{\log N/N} + K_N^{-k_o}$,

$$
\begin{aligned}
K_N\sqrt{\log N}\mathfrak{f}_*^{-(q-1/2)}K_N\rho_N &\le CK_N^{9/2}(\log N)^4 N^{-1/2} + CK_N^{7/2-k_o}(\log N)^{7/2} \\
&= C\left(\frac{K_N^9(\log N)^8}{N}\right)^{1/2} + CK_N^{7/2-k_o}(\log N)^{7/2}, \\
K_N\sqrt{\log N}\mathfrak{f}_*^{(q-2)/2}K_N^{-1/2} &= (\log N)^{-1/2}, \\
K_N\sqrt{\log N}\mathfrak{f}_*^{1/2}K_N^{-(q-2)/2} &\le K_N^{(4-q)/2}\sqrt{\log N} \le K_N^{-1/2}\sqrt{\log N}, \\
K_N\sqrt{\log N}K_N^{1/2}\rho_N &\le K_N^{5/2}\log N/\sqrt{N} + K_N^{3/2-k_o}\sqrt{\log N}, \\
K_N\sqrt{\log N}K_N^{-(k_o-1)} &= K_N^{2-k_o}\sqrt{\log N},
\end{aligned}
$$

and every right side vanishes by Assumption 5(a), which also gives $k_o > 7/2$ and hence $q \ge 5$, and by the polynomial growth of K_N in N. Therefore $K_N\sqrt{\log N}\sigma_N \to 0$.

Step 2 (Claim (i)): By the Cauchy–Schwarz inequality and $P_o^t \ge \kappa$, uniformly over unit vectors v,

$$
\left|\langle \widehat{\Delta P}, \mathbb{1}_{\mathcal{B}_N}\widehat{\Delta D}V'\cdot\rangle_{P_o}\right| \le C\kappa^{-1}R_W\|\widehat{\Delta P}\|_{L^2(\mathbb{P})}\sigma_N \le C\rho_N\sigma_N,
$$

and $\sqrt{N}\rho_N\sigma_N \le CK_N\sqrt{\log N}\sigma_N \to 0$ by Step 1, since $\rho_N = O(r_N)$.

Step 3 (Claim (ii)): The class $\{\ell[\mathbb{1}_{\mathcal{B}_N}\widehat{\Delta D}V'\cdot]\}$, indexed by $(\hat{\boldsymbol{\gamma}}, \hat{\xi}, v)$ over $\mathcal{N} \times (\Xi(K_N) \cap \mathcal{U}_N) \times \{|||V||| = 1\}$, has the structure of Lemma C.4 with $\psi(w) = \mathbb{1}_{\mathcal{B}_N}(w)V'w$, constant envelope $F_N = C$ by Lemma B.3(i), and variance parameter $C\sigma_N$ by Step 1. Lemma C.4 gives $O_{\mathrm{P}}(\sigma_N K_N\sqrt{\log N/N} + K_N^2\log N/N)$, and both terms times $\sqrt{N}$ vanish, the first by Step 1 and the second by Assumption 5(a). □

Lemma C.12 (comparison expansion)**.** *There are random vectors $R_N(v)$ and a random matrix E_N on $\mathcal{T}_{\boldsymbol{\gamma}_o}$ with $\sup_{\|v\|=1}|R_N(v)| = o_{\mathrm{P}}(N^{-1/2})$ and $\|E_N\|_{\mathrm{op}} = o_{\mathrm{P}}(1)$ such that, for every unit vector $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$,*

$$
v'(I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} + E_N)\Pi_o\mathrm{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) = (\mathbb{P}_N - \mathbb{P})\ell[\phi^{\mathrm{eff}}_{N,v}] + R_N(v),
$$

where $\Pi_o := \mathbf{I}_L - \mathrm{vec}(\boldsymbol{\gamma}_o)\mathrm{vec}(\boldsymbol{\gamma}_o)'$ is the projection onto $\mathcal{T}_{\boldsymbol{\gamma}_o}$.

Proof. All statements are conditional on the event $\hat{\theta} \in \mathcal{U}_N$, of probability tending to one.

Step 1 (Comparison inequalities): Optimality of $\hat{\theta}$ over $\Gamma \times \Xi(K_N)$ and feasibility of the

paths give, for both σ and every $\varepsilon \in (0, N^{-1/2}(\log N)^{-1}]$,

$$0 \le \mathcal{L}_N(\hat{\theta}) - \mathcal{L}_N(\theta_\varepsilon^\sigma) = -(\mathbb{P}_N - \mathbb{P})[l(\cdot, \theta_\varepsilon^\sigma) - l(\cdot, \hat{\theta})] - \mathbb{P}[l(\cdot, \theta_\varepsilon^\sigma) - l(\cdot, \hat{\theta})].$$

Step 2 (Population part): Along the path, $\varepsilon \mapsto \mathbb{P}l(\cdot, \theta_\varepsilon^\sigma)$ is differentiable with derivative $\mathbb{E}\sum_t T^t \dot{e}_s^{\sigma,t} / P_{\xi_s^\sigma}^t(\boldsymbol{\gamma}_s^{\sigma\prime} W)$ at $\varepsilon = s$, where $\dot{e}_s^\sigma$ is the path derivative of the fitted values at s. Lemma C.8 turns the derivative into $-\langle P_{\xi_s^\sigma}(\boldsymbol{\gamma}_s^{\sigma\prime}\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), \dot{e}_s^\sigma\rangle_{P_o} + O(\|\dot{e}\|_\infty \rho_N^2)$. The path is Lipschitz in s with $\|P_{\xi_s^\sigma}(\boldsymbol{\gamma}_s^{\sigma\prime}\cdot) - P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot)\|_{L^2(p_W)} \le Cs$ and $\|\dot{e}_s^\sigma - \dot{e}_0^\sigma\|_\infty \le CK_N s$, the latter by Lemma B.3(ii), so integrating over $s \in [0, \varepsilon]$,

$$\mathbb{P}[l(\cdot, \theta_\varepsilon^\sigma) - l(\cdot, \hat{\theta})] = -\varepsilon\langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), \dot{e}_0^\sigma\rangle_{P_o} + O(\varepsilon^2(1 + K_N\rho_N) + \varepsilon\rho_N^2),$$

and $\varepsilon \le N^{-1/2}(\log N)^{-1} = o(N^{-1/2})$ with $\rho_N^2 = o(N^{-1/2})$ make the remainder $\varepsilon \cdot o_{\mathrm{P}}(N^{-1/2})$.

Step 3 (Dividing and combining the signs): Substituting Step 2 and Lemma C.9(i) into Step 1 and dividing by ε,

$$\begin{aligned} 0 \le\ & -\sigma[(\mathbb{P}_N - \mathbb{P})\ell[\hat{d}_v] - \langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), \hat{d}_v\rangle_{P_o}] \\ & - \left[(\mathbb{P}_N - \mathbb{P})\ell[\widehat{\Delta P}^\circ] - \langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), \widehat{\Delta P}^\circ\rangle_{P_o}\right] + o_{\mathrm{P}}(N^{-1/2}), \end{aligned}$$

using $\dot{e}_0^\sigma = \sigma\hat{d}_v + \widehat{\Delta P}^\circ$. The bracket in $\widehat{\Delta P}^\circ$ is $o_{\mathrm{P}}(N^{-1/2})$: the empirical part by Lemma C.9(ii), and the pairing by $|\langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), \widehat{\Delta P}^\circ\rangle_{P_o}| \le C\rho_N \cdot \rho_N/\bar{\varepsilon} = O_{\mathrm{P}}(\rho_N^2)$. Taking $\sigma = +1$ and $\sigma = -1$ in turn, by the sandwich argument,

$$\langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot), \hat{d}_v\rangle_{P_o} = (\mathbb{P}_N - \mathbb{P})\ell[\hat{d}_v] + o_{\mathrm{P}}(N^{-1/2}), \qquad \textit{uniformly over } v. \tag{C.3}$$

Step 4 (Empirical side of (C.3)): By Lemma C.9(iii) and Lemma C.11(ii), $(\mathbb{P}_N - \mathbb{P})\ell[\hat{d}_v] = (\mathbb{P}_N - \mathbb{P})\ell[\phi_{N,v}^{\mathrm{eff}}] + o_{\mathrm{P}}(N^{-1/2})$.

Step 5 (Pairing side of (C.3)): Decompose $P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'w) - P_o(\boldsymbol{\gamma}_o'w) = DP_o(\boldsymbol{\gamma}_o'w)(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)'w + \widehat{\Delta P}(\boldsymbol{\gamma}_o'w) + \hat{q}(w)$, where $\hat{q}(w) := (\int_0^1 [DP_{\hat{\xi}}(\boldsymbol{\gamma}_u'w) - DP_o(\boldsymbol{\gamma}_o'w)]du)(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)'w$ along $\boldsymbol{\gamma}_u := \boldsymbol{\gamma}_o + u(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$. Write $\hat{d}_v = \phi_{N,v}^{\mathrm{eff}} + \Delta d$, $\Delta d := [DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'w)\hat{V}'w - DP_o(\boldsymbol{\gamma}_o'w)V'w] + [\nabla(b_{K_N}'a_N(v))(\boldsymbol{\gamma}_o'w) - \nabla(b_{K_N}'a_N(v))(\hat{\boldsymbol{\gamma}}'w)]$. The pairing splits into six terms.

(a) $\langle DP_o(\boldsymbol{\gamma}_o'\cdot)(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)'\cdot,\ \phi_{N,v}^{\mathrm{eff}}\rangle_{P_o} = \mathrm{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)'[I_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\mathrm{eff}} v + O(K_N^{-(1+\alpha)})]$: for any $U \in \mathbb{R}^{\dim(W)\times 2}$, $\langle \phi_U, \phi_v^{\mathrm{eff}}\rangle_{P_o} = \langle \phi_U - h_U^{\mathrm{lf}}, \phi_v^{\mathrm{eff}}\rangle_{P_o} = \mathrm{vec}(U)' I_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\mathrm{eff}} v$ since $h_U^{\mathrm{lf}} \in \mathcal{T} \perp \phi_v^{\mathrm{eff}}$, and replacing ϕ_v^{eff} by $\phi_{N,v}^{\mathrm{eff}}$ costs $\|\phi_{N,v}^{\mathrm{eff}} - \phi_v^{\mathrm{eff}}\|_{P_o} \le CK_N^{-(1+\alpha)}$ per unit of $|||U|||$.

(b) $\langle \widehat{\Delta P}, \phi^{\text{eff}}_{N,v} \rangle_{P_o} = \langle \widehat{\Delta P}, \phi^{\text{eff}}_{N,v} - \phi^{\text{eff}}_{v} \rangle_{P_o} = O(\rho_N K_N^{-(1+\alpha)})$, since $\widehat{\Delta P}$ is a gradient field in $\mathcal{T}$ and $\phi^{\text{eff}}_v \perp \mathcal{T}$. $\sqrt{N}\rho_N K_N^{-(1+\alpha)} \to 0$, so this is $o_{\text{P}}(N^{-1/2})$.

(c) $\langle \hat{q}, \hat{d}_v \rangle_{P_o} \le C|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| \, \mathbb{E}\| \int_0^1 [DP_{\hat{\xi}}(\boldsymbol{\gamma}_u' W) - DP_o(\boldsymbol{\gamma}_o' W)] du \| \le C|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||(K_N \rho_N + h_N)$ by Lemma C.5(ii), an $o_{\text{P}}(1)$ multiple of $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||$.

(d) $\langle DP_o(\boldsymbol{\gamma}_o' \cdot)(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)' \cdot, \ \Delta d \rangle_{P_o} \le C|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||(\mathbb{E}\|\widehat{\Delta D}\| + C\rho_N) \le C|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||(K_N \rho_N + h_N + \rho_N)$, again an $o_{\text{P}}(1)$ multiple of $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||$.

(e) $\langle \widehat{\Delta P}, \mathbb{1}_{\mathcal{B}_N^c} \Delta d \rangle_{P_o} \le C\rho_N(K_N \rho_N + \rho_N) = o_{\text{P}}(N^{-1/2})$, using the core rates of Lemma C.5 in L^2 off the layer and the $C\rho_N$ bounds on the two non-derivative parts of Δd.

(f) $\langle \widehat{\Delta P}, \mathbb{1}_{\mathcal{B}_N} \widehat{\Delta D} V' \cdot \rangle_{P_o} = o_{\text{P}}(N^{-1/2})$ by Lemma C.11(i). The remaining layer parts of Δd are bounded by $C\rho_N$ pointwise and are covered by (e).

Finally, $\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$ has exactly zero intercept-mean components, and its radial component has norm $\frac{1}{2}|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||^2 = O_{\text{P}}(\rho_N^2)$ by the sphere normalization, so replacing $\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$ by $\Pi_o \text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$ in (a) costs $O_{\text{P}}(\rho_N^2) = o_{\text{P}}(N^{-1/2})$.

Step 6 (Collection): Steps 3–5 give, uniformly over v,

$$v' I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} \Pi_o \text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) + o_{\text{P}}(1) \, \|\Pi_o \text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)\| = (\mathbb{P}_N - \mathbb{P}) \ell[\phi^{\text{eff}}_{N,v}] + o_{\text{P}}(N^{-1/2}),$$

where the $o_{\text{P}}(1)$ collects the coefficients of $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||$ in (c), (d) and the $O(K_N^{-(1+\alpha)})$ of (a), all uniform in v. This is the display with E_N the induced matrix perturbation. □

Lemma C.13 (central limit theorem for the direction scores)**.** *Let $v_1, \ldots, v_r$ be an orthonormal basis of $\mathcal{T}_{\boldsymbol{\gamma}_o}$ and $\zeta_{N,l}(\omega) := \ell[\phi^{\text{eff}}_{N,v_l}](\omega)$. Then $\mathbb{E}\zeta_{N,l} = 0$, $\mathbb{E}[\zeta_{N,l}\zeta_{N,m}] = \langle \phi^{\text{eff}}_{N,v_l}, \phi^{\text{eff}}_{N,v_m} \rangle_{P_o} = v_l' I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v_m + O(K_N^{-(1+\alpha)})$, and*

$$(\sqrt{N}\mathbb{P}_N \zeta_{N,l})_{l=1}^r \to_d \mathcal{N}(0, \ (v_l' I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v_m)_{l,m}).$$

Moreover, $\mathbb{E}[(\zeta_{N,l} - \ell[\phi^{\text{eff}}_{v_l}])^2] = \|\phi^{\text{eff}}_{N,v_l} - \phi^{\text{eff}}_{v_l}\|^2_{P_o} \le C K_N^{-2(1+\alpha)}$, and $\ell[\phi^{\text{eff}}_{v_l}] = v_l' s^{\text{eff}}_{\boldsymbol{\gamma}}$ is the efficient score in direction v_l.

Proof. The mean is zero by Lemma C.8 at $\theta = \theta_o$. For the second moments, mutual exclusivity of the treatment indicators gives, for any two triples $u, \tilde{u}$ with $\sum_t u^t = \sum_t \tilde{u}^t = 0$, $\mathbb{E}[\ell[u]\ell[\tilde{u}]] = \mathbb{E}\sum_t P_o^t(Z) \frac{u^t \tilde{u}^t}{(P_o^t(Z))^2} = \langle u, \tilde{u} \rangle_{P_o}$, and $\langle \phi^{\text{eff}}_{N,v_l}, \phi^{\text{eff}}_{N,v_m} \rangle_{P_o} = \langle \phi^{\text{eff}}_{v_l}, \phi^{\text{eff}}_{v_m} \rangle_{P_o} +$

$O(K_N^{-(1+\alpha)}) = v_l' I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} v_m + O(K_N^{-(1+\alpha)})$ by Lemma C.3(i). The summands are bounded uniformly in N by $\kappa^{-1} \sup_N \|\phi^{\text{eff}}_{N,v}\|_\infty < \infty$, so the Lindeberg condition holds trivially for the triangular array, and the joint convergence follows from the Cramér–Wold device. The last display is the same second-moment identity applied to $\phi^{\text{eff}}_{N,v} - \phi^{\text{eff}}_v$, and $\ell[\phi^{\text{eff}}_v] = v' s^{\text{eff}}_{\boldsymbol{\gamma}}$ by the definition of the efficient score. □

Proof of Theorem 3. Let $v_1, \ldots, v_r$ be an orthonormal basis of $\mathcal{T}_{\boldsymbol{\gamma}_o}$ and stack the displays of Lemma C.12:

$$(I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} + E_N)\Pi_o \sqrt{N}\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) = (\sqrt{N}(\mathbb{P}_N - \mathbb{P})\ell[\phi^{\text{eff}}_{N,v_l}])_{l=1}^r + o_{\text{P}}(1),$$

read as an equation on $\mathcal{T}_{\boldsymbol{\gamma}_o}$ in the basis coordinates. The right side is $O_{\text{P}}(1)$ by Lemma C.13. By Lemma C.1, $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$ is positive definite on $\mathcal{T}_{\boldsymbol{\gamma}_o}$, so $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} + E_N$ is invertible on $\mathcal{T}_{\boldsymbol{\gamma}_o}$ with bounded inverse with probability tending to one, and $\sqrt{N}\Pi_o \text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) = O_{\text{P}}(1)$. The term $E_N \Pi_o \sqrt{N}\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$ is then $o_{\text{P}}(1)$, and

$$I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} \Pi_o \sqrt{N}\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) = \left(\sqrt{N}\mathbb{P}_N \ell[\phi^{\text{eff}}_{N,v_l}]\right)_{l=1}^r + o_{\text{P}}(1),$$

using $\mathbb{E}\ell[\phi^{\text{eff}}_{N,v}] = 0$ to drop the centering. By the last part of Lemma C.13, replacing $\ell[\phi^{\text{eff}}_{N,v_l}]$ by the efficient score $v_l' s^{\text{eff}}_{\boldsymbol{\gamma}}$ changes the right side by a mean-zero average with variance $O(K_N^{-2(1+\alpha)}) \to 0$, hence by $o_{\text{P}}(1)$. The intercept-mean components of $\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)$ vanish exactly by the normalization of Γ, and the radial component has norm $\frac{1}{2}|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||^2\sqrt{N} = O_{\text{P}}(\sqrt{N}\rho_N^2) = o_{\text{P}}(1)$, so $\sqrt{N}\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) = \Pi_o\sqrt{N}\text{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) + o_{\text{P}}(1)$. Inverting $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$ on $\mathcal{T}_{\boldsymbol{\gamma}_o}$ through its Moore–Penrose inverse and applying Slutsky,

$$\sqrt{N}\left(\text{vec}(\hat{\boldsymbol{\gamma}}) - \text{vec}(\boldsymbol{\gamma}_o)\right) = \left(I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}\right)^+ \frac{1}{\sqrt{N}} \sum_{i=1}^N s^{\text{eff}}_{\boldsymbol{\gamma}}(T_i, W_i) + o_{\text{P}}(1) \to_d \mathcal{N}(0, (I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+),$$

which is the asymptotically linear representation of the theorem, and efficiency in the sense of Definition 2 follows by Proposition 2. □

C.1 Regular Functionals of the First Stage

Theorem 3 concerns the index coefficients alone. This subsection states and proves the asymptotic distribution of functionals of the full first-stage parameter $\theta = (\boldsymbol{\gamma}, P)$ that can be estimated at the parametric rate. For a propensity perturbation $u = (u^1, u^2)$ on $\mathcal{Z}$, let $s_P[u] := \sum_{t\in\{0,1,2\}} T^t u^t(Z)/P_o^t(Z)$ with $u^0 := -(u^1 + u^2)$ denoting the score of the

path $P_o + tu$ at $t = 0$ holding $\boldsymbol{\gamma}_o$ fixed. The tangent space of the first-stage model at θ_o, denoted $\mathcal{T}_{\theta_o}$, is the closed linear span in $L^2(\mathbb{P})$ of the scores $v's_{\boldsymbol{\gamma}} + s_P[\nabla\delta G]$, with $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ and δG ranging over the smooth functions on $\bar{\mathcal{Z}}$, the perturbations of the social surplus function at $\boldsymbol{\gamma}_o$.

Definition 3. A functional $\Lambda : \Gamma \times \mathcal{P} \to \mathbb{R}$ is *regular* if there exist a matrix $\Lambda_{\boldsymbol{\gamma}} \in \mathbb{R}^{\dim(W)\times 2}$ and a vector field $\Lambda_P = (\Lambda_P^1, \Lambda_P^2)$ of class $C^{1,\alpha}(\mathcal{Z})$, with α as in Assumption 5(b), such that:

(i) For every $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ and $\delta G \in C^{2,\alpha}(\mathcal{Z})$, with $\boldsymbol{\gamma}_t := (\boldsymbol{\gamma}_o + tV)/|||\boldsymbol{\gamma}_o + tV|||$,

$$\frac{d}{dt}\Lambda\left(\boldsymbol{\gamma}_t, P_o + t\nabla\delta G\right)\Big|_{t=0} = \langle \Lambda_{\boldsymbol{\gamma}}, V\rangle + \int_{\mathcal{Z}} \Lambda_P(z)'\nabla\delta G(z)dz.$$

(ii) The remainder $R_\Lambda(\theta) := \Lambda(\theta) - \Lambda(\theta_o) - \langle \Lambda_{\boldsymbol{\gamma}}, \boldsymbol{\gamma} - \boldsymbol{\gamma}_o\rangle - \int_{\mathcal{Z}} \Lambda_P(z)'(P - P_o)(z)dz$ satisfies, for every $C > 0$,

$$\sup\left\{\frac{|R_\Lambda(\theta)|}{N^{-1/2} + |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o|||} : \theta = (\boldsymbol{\gamma}, P_\xi),\ \xi \in \Xi(K_N),\ |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| \le C\rho_N,\ \mathsf{L}(\theta, \theta_o) \le C\rho_N\right\} \to 0.$$

Condition (i) is pathwise differentiability along the submodels of the first stage, and Condition (ii) is the corresponding expansion at sieve elements near θ_o, with a remainder negligible relative to the parametric rate and to the index error. For a regular functional, the derivative of Λ along the score $v's_{\boldsymbol{\gamma}} + s_P[\nabla\delta G]$ of the direction $(v, \nabla\delta G)$, with $\delta G \in C^\infty(\bar{\mathcal{Z}})$, is

$$\dot{\Lambda}(v's_{\boldsymbol{\gamma}} + s_P[\nabla\delta G]) := \langle \Lambda_{\boldsymbol{\gamma}}, V\rangle + \int_{\mathcal{Z}} \Lambda_P(z)'\nabla\delta G(z)dz,$$

where V is the matrix form of v and $\langle A, B\rangle = \mathrm{vec}(A)'\mathrm{vec}(B)$. In the notation of this section, the score is $v's_{\boldsymbol{\gamma}} + s_P[\nabla\delta G] = \ell[\phi_v + \nabla\delta G]$. The functional $\dot{\Lambda}$ extends to a bounded linear functional on $\mathcal{T}_{\theta_o}$ (Lemma C.14), and its Riesz representer $\tilde{\psi}_\Lambda \in \mathcal{T}_{\theta_o}$, the element with $\mathbb{E}[\tilde{\psi}_\Lambda s] = \dot{\Lambda}(s)$ for all $s \in \mathcal{T}_{\theta_o}$, is the efficient influence function of Λ.

Theorem C.1. *Let Λ be a regular functional. Under Assumptions 1, 3 and 4–5,*

$$\sqrt{N}(\Lambda(\hat{\theta}) - \Lambda(\theta_o)) = \frac{1}{\sqrt{N}}\sum_{i=1}^N \tilde{\psi}_\Lambda(T_i, W_i) + o_{\mathrm{P}}(1) \to_d \mathcal{N}(0, \sigma_\Lambda^2),\ \text{where } \sigma_\Lambda^2 = \mathbb{E}[\tilde{\psi}_\Lambda(T_i, W_i)^2],$$

and the representation holds jointly for finitely many regular functionals. Moreover, σ_Λ^2 is the semiparametric efficiency bound for $\Lambda(\theta_o)$ in the first-stage model.

Lemma C.14 below provides $\tilde{\psi}_\Lambda$ and σ^2_Λ in closed form.

For a regular functional Λ, let

$$f_\Lambda := (p_Z \mathbb{W})^{-1} \Lambda_P, \qquad h_\Lambda := \Pi_\infty f_\Lambda.$$

The field f_Λ represents the propensity derivative $u \mapsto \int_{\mathcal{Z}} \Lambda_P(z)' u(z) dz$ of Λ on all $\mathbb{R}^2$-valued fields of z, and h_Λ represents it on $\mathcal{T}$ (Lemma C.14(i)). The field h_Λ is the propensity representer of Λ. By Lemma C.2(i), $h_\Lambda = \nabla \delta G_\Lambda$ for a surplus perturbation $\delta G_\Lambda \in H^1(\operatorname{Int} \mathcal{Z})$, unique up to an additive constant.

Already for $\Lambda_P \in L^2(\mathcal{Z})$, the Riesz representation theorem gives h_Λ and a finite efficiency bound. A finite bound alone does not ensure estimation at the parametric rate. Ritov and Bickel (1990) show that the integrated squared density has a finite efficiency bound, while no estimator converges at a polynomial rate when the densities are not restricted in smoothness. Theorem C.1 uses in addition the smoothness $\delta G_\Lambda \in C^{2,\alpha}(\mathcal{Z})$, which Lemma C.14(i) derives from $\Lambda_P \in C^{1,\alpha}(\mathcal{Z})$ and which the spline approximation of h_Λ in Lemma C.14(iii) requires.

Along the least favorable submodel $t \mapsto (\boldsymbol{\gamma}_t, P_o - t h^{\mathrm{lf}}_V)$ of an index direction $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, Definition 3(i) gives Λ the derivative $\langle \Lambda_{\boldsymbol{\gamma}}, V \rangle - \int_{\mathcal{Z}} \Lambda_P(z)' h^{\mathrm{lf}}_V(z) dz$. The integral is linear in V, and Lemma C.14(ii) below writes it as $\langle \Lambda^{\mathrm{lf}}, V \rangle$ for a matrix $\Lambda^{\mathrm{lf}} \in \mathbb{R}^{\dim(W) \times 2}$. The derivative of Λ along the least favorable submodel is then $\langle \Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}}, V \rangle$.

Lemma C.14 (representer of a regular functional)**.** *Let Λ be a regular functional. Under Assumptions 1, 3, 4 and 5:*

(i) $\delta G_\Lambda \in C^{2,\alpha}(\mathcal{Z})$*, and* $\int_{\mathcal{Z}} \Lambda_P(z)' u(z) dz = \langle u, h_\Lambda \rangle_{P_o}$ *for every* $u \in \mathcal{T}$*.*

(ii) There is a unique matrix $\Lambda^{\mathrm{lf}} \in \mathbb{R}^{\dim(W) \times 2}$ *such that, for every* $V \in \mathbb{R}^{\dim(W) \times 2}$*,*

$$\langle \Lambda^{\mathrm{lf}}, V \rangle = \int_{\mathcal{Z}} \Lambda_P(z)' h^{\mathrm{lf}}_V(z) dz = \langle \phi_V, h_\Lambda \rangle_{P_o}.$$

(iii) Let $a_N(\Lambda) \in \mathbb{R}^{d_{K_N}}$ *be the coefficients* $Q_{K_N} \widetilde{\delta G}_\Lambda$ *of the quasi-interpolant of Lemma B.4 applied to an extension* $\widetilde{\delta G}_\Lambda \in C^{2,\alpha}(\bar{\mathcal{Z}})$ *of* δG_Λ *to the knot box as in Lemma C.3; then* $\|\nabla(b'_{K_N} a_N(\Lambda)) - h_\Lambda\|_{L^\infty(\mathcal{Z})} \le C K_N^{-(1+\alpha)}$*,* $\|\nabla(b'_{K_N} a_N(\Lambda))\|_{L^\infty} + \|D^2(b'_{K_N} a_N(\Lambda))\|_{L^\infty} \le C$*, and there are* $\bar{\varepsilon}_\Lambda > 0$ *and* N_Λ *such that* $\xi^\circ \pm \bar{\varepsilon} a_N(\Lambda) \in \Xi(K_N)$ *for all* $N \ge N_\Lambda$ *and all* $\bar{\varepsilon} \in (0, \bar{\varepsilon}_\Lambda]$*.*

(iv) Let V_Λ be the matrix with $\mathrm{vec}(V_\Lambda) = v_\Lambda := (I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^{+}\mathrm{vec}(\Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}})$, and let $r_\Lambda := h_\Lambda + \phi^{\mathrm{eff}}_{v_\Lambda}$, where $\phi^{\mathrm{eff}}_{v_\Lambda} = \phi_{V_\Lambda} - h^{\mathrm{lf}}_{V_\Lambda}$ is the efficient image of v_Λ. Then $\ell[r_\Lambda] \in \mathcal{T}_{\theta_o}$, the functional $\dot\Lambda$ is bounded on $\mathcal{T}_{\theta_o}$ with $\mathbb{E}[\ell[r_\Lambda]s] = \dot\Lambda(s)$ for every $s \in \mathcal{T}_{\theta_o}$, so that $\tilde\psi_\Lambda = \ell[r_\Lambda]$, and

$$\sigma^2_\Lambda = \mathbb{E}[\ell[r_\Lambda]^2] = \|h_\Lambda\|^2_{P_o} + \mathrm{vec}(\Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}})' \left(I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}\right)^{+} \mathrm{vec}(\Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}}).$$

Proof. (i) Under Assumption 5(b), f_Λ is an $\mathbb{R}^2$-valued field of class $C^{1,\alpha}(\mathcal{Z})$, since p_Z is bounded away from zero on $\mathcal{Z}$ by Assumption 4(f) and $\mathbb{W}$ is of class C^{k_o}. By the identity $\langle u, \tilde u\rangle_{P_o} = \int_{\mathcal{Z}} u(z)'\mathbb{W}(z)\tilde u(z)p_Z(z)dz$ for $\mathbb{R}^2$-valued fields, every $\mathbb{R}^2$-valued field u of z satisfies

$$\int_{\mathcal{Z}} \Lambda_P(z)'u(z)dz = \langle u, f_\Lambda\rangle_{P_o}. \tag{C.4}$$

The proof of Lemma C.2 uses f_v only through its membership in $C^{1,\alpha}(\mathcal{Z})$ and applies verbatim to f_Λ. The potential δG_Λ of $h_\Lambda = \Pi_\infty f_\Lambda$ is characterized by (C.1) with $(\delta G_v, f_v)$ replaced by $(\delta G_\Lambda, f_\Lambda)$, and the proof of Lemma C.2(iii) gives $\delta G_\Lambda \in C^{2,\alpha}(\mathcal{Z})$. Since $h_\Lambda - f_\Lambda \perp \mathcal{T}$, $\langle u, h_\Lambda\rangle_{P_o} = \langle u, f_\Lambda\rangle_{P_o}$ for $u \in \mathcal{T}$, which is the second claim by (C.4).

(ii) The map $V \mapsto h^{\mathrm{lf}}_V$ is linear, and so is $V \mapsto \int_{\mathcal{Z}} \Lambda_P(z)'h^{\mathrm{lf}}_V(z)dz$ on $\mathbb{R}^{\dim(W)\times 2}$. Every linear functional on $\mathbb{R}^{\dim(W)\times 2}$ is $V \mapsto \langle A, V\rangle = \mathrm{vec}(A)'\mathrm{vec}(V)$ for a unique matrix A, which gives Λ^{lf} and the first equality. For the second, $\phi_V - h^{\mathrm{lf}}_V \perp \mathcal{T}$ and $h_\Lambda \in \mathcal{T}$ give $\langle \phi_V, h_\Lambda\rangle_{P_o} = \langle h^{\mathrm{lf}}_V, h_\Lambda\rangle_{P_o}$, which equals $\int_{\mathcal{Z}} \Lambda_P(z)'h^{\mathrm{lf}}_V(z)dz$ by (i).

(iii) Lemmas C.3 and C.7 use δG_v only through the bound on $\|\delta G_v\|_{C^{2,\alpha}(\mathcal{Z})}$ and apply verbatim to δG_Λ.

(iv) Write $r_\Lambda = \phi_{v_\Lambda} + u_\Lambda$ with $u_\Lambda := h_\Lambda - h^{\mathrm{lf}}_{v_\Lambda} \in \mathcal{T}$, so that $\ell[r_\Lambda] \in \mathcal{T}_{\theta_o}$. For $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ and $u = \nabla\delta G$ with $\delta G \in C^\infty(\bar{\mathcal{Z}})$, the second-moment identity in the proof of Lemma C.13 gives $\mathbb{E}[\ell[r_\Lambda]\ell[\phi_v + u]] = \langle r_\Lambda, \phi_v + u\rangle_{P_o}$. Expanding, $\langle h_\Lambda, \phi_v\rangle_{P_o} = \langle \Lambda^{\mathrm{lf}}, V\rangle$ by (ii), $\langle h_\Lambda, u\rangle_{P_o} = \int_{\mathcal{Z}} \Lambda_P(z)'u(z)dz$ by (i), $\langle \phi^{\mathrm{eff}}_{v_\Lambda}, u\rangle_{P_o} = 0$, and $\langle \phi^{\mathrm{eff}}_{v_\Lambda}, \phi_v\rangle_{P_o} = \langle \phi^{\mathrm{eff}}_{v_\Lambda}, \phi^{\mathrm{eff}}_v\rangle_{P_o} = v'I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}v_\Lambda = v'\Pi_o\mathrm{vec}(\Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}}) = \langle \Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}}, V\rangle$, using $h^{\mathrm{lf}}_v \perp \phi^{\mathrm{eff}}_{v_\Lambda}$, the identity $I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}(I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^{+} = \Pi_o$ from Lemma C.1, and $v \in \mathcal{T}_{\boldsymbol{\gamma}_o}$. The sum is $\dot\Lambda(\ell[\phi_v + u])$. Hence $|\dot\Lambda(s)| \le \|\ell[r_\Lambda]\|_{L^2(\mathbb{P})}\|s\|_{L^2(\mathbb{P})}$ on the span, $\dot\Lambda$ extends by continuity to $\mathcal{T}_{\theta_o}$, and $\ell[r_\Lambda]$ is its Riesz representer. Finally, $\mathbb{E}[\ell[r_\Lambda]^2] = \|r_\Lambda\|^2_{P_o} = \|h_\Lambda\|^2_{P_o} + \|\phi^{\mathrm{eff}}_{v_\Lambda}\|^2_{P_o}$ since $\phi^{\mathrm{eff}}_{v_\Lambda} \perp \mathcal{T}$, and $\|\phi^{\mathrm{eff}}_{v_\Lambda}\|^2_{P_o} = v'_\Lambda I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}v_\Lambda$ is the stated quadratic form. □

Lemma C.15 (comparison expansion along a propensity direction)**.** *Let Λ be a regular functional, let $a_N(\Lambda)$ be the coefficients of the quasi-interpolant of δG_Λ in Lemma C.14(iii),*

and let $\widehat{\Delta P}(z) := P_{\hat{\xi}}(z) - P_o(z)$ *as in Lemma C.11. Then*

$$\int_{\mathcal{Z}} \Lambda_P(z)' \widehat{\Delta P}(z) dz + \langle \Lambda^{\mathrm{lf}}, \hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o \rangle$$
$$= (\mathbb{P}_N - \mathbb{P}) \ell[\nabla(b'_{K_N} a_N(\Lambda))(\boldsymbol{\gamma}'_o \cdot)] + o_{\mathrm{P}}(N^{-1/2}) + o_{\mathrm{P}}(1) |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||.$$

Proof. All statements are conditional on the event $\hat{\theta} \in \mathcal{U}_N$, of probability tending to one. Let $\bar{\varepsilon} := \bar{\varepsilon}_\Lambda$ from Lemma C.14(iii), and for $\sigma \in \{+1, -1\}$ and $\varepsilon \in (0, N^{-1/2}(\log N)^{-1}]$ define

$$\xi^\sigma_\varepsilon := (1 - \varepsilon/\bar{\varepsilon})\, \hat{\xi} + (\varepsilon/\bar{\varepsilon})\, (\xi^\circ + \sigma \bar{\varepsilon} a_N(\Lambda)), \qquad \theta^\sigma_\varepsilon := (\hat{\boldsymbol{\gamma}}, P_{\xi^\sigma_\varepsilon}).$$

The path is feasible, since ξ^σ_ε is a convex combination of $\hat{\xi}$ and the point $\xi^\circ + \sigma \bar{\varepsilon} a_N(\Lambda) \in \Xi(K_N)$, and it lies in $\mathcal{U}_N$ for C_U enlarged, since the fitted values move by at most $C\varepsilon$. The index is fixed along the path and the fitted propensity is affine in ε: $P_{\xi^\sigma_\varepsilon}(\hat{\boldsymbol{\gamma}}' w) - P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}' w) = \varepsilon \dot{e}^\sigma(w)$ with $\dot{e}^\sigma := \sigma \hat{d}_\Lambda + \widehat{\Delta P}^\circ$, where $\hat{d}_\Lambda(w) := \nabla(b'_{K_N} a_N(\Lambda))(\hat{\boldsymbol{\gamma}}' w)$ and $\widehat{\Delta P}^\circ$ is the field defined before Lemma C.9.

Step 1 (Pinned pairing): Optimality of $\hat{\theta}$ and feasibility of the path give $0 \le \mathcal{L}_N(\hat{\theta}) - \mathcal{L}_N(\theta^\sigma_\varepsilon)$, as in Step 1 of Lemma C.12. Step 2 of that proof applies with $\dot{e}^\sigma_s = \dot{e}^\sigma$ for every s, so $\mathbb{P}[l(\cdot, \theta^\sigma_\varepsilon) - l(\cdot, \hat{\theta})] = -\varepsilon \langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}' \cdot) - P_o(\boldsymbol{\gamma}'_o \cdot), \dot{e}^\sigma \rangle_{P_o} + \varepsilon \cdot o_{\mathrm{P}}(N^{-1/2})$. Lemma C.9(i) applies with $g_2 = 0$, since the fitted values are affine in ε, and with the bounds on g_1 and g_3 unchanged. Lemma C.9(ii) is unchanged. Dividing by ε and combining the two signs as in Step 3 of Lemma C.12,

$$\langle P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}' \cdot) - P_o(\boldsymbol{\gamma}'_o \cdot), \hat{d}_\Lambda \rangle_{P_o} = (\mathbb{P}_N - \mathbb{P}) \ell[\hat{d}_\Lambda] + o_{\mathrm{P}}(N^{-1/2}). \tag{C.5}$$

Step 2 (Empirical side of (C.5)): By Lemma C.14(iii),

$$\left\| \hat{d}_\Lambda - \nabla(b'_{K_N} a_N(\Lambda))(\boldsymbol{\gamma}'_o \cdot) \right\|_\infty \le \| D^2(b'_{K_N} a_N(\Lambda)) \|_\infty R_W |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| \le C \rho_N.$$

The class $\{\ell[\nabla(b'_{K_N} a_N(\Lambda))(\boldsymbol{\gamma}' \cdot) - \nabla(b'_{K_N} a_N(\Lambda))(\boldsymbol{\gamma}'_o \cdot)] : \boldsymbol{\gamma} \in \mathcal{N}\}$ is indexed by the fixed-dimensional set $\mathcal{N}$, with envelope and variance parameter $C\rho_N$. Lemma C.4 therefore gives

$$(\mathbb{P}_N - \mathbb{P}) \ell[\hat{d}_\Lambda] = (\mathbb{P}_N - \mathbb{P}) \ell[\nabla(b'_{K_N} a_N(\Lambda))(\boldsymbol{\gamma}'_o \cdot)] + O_{\mathrm{P}} \left(\rho_N K_N \sqrt{\frac{\log N}{N}} + \rho_N K_N^2 \frac{\log N}{N} \right),$$

and the remainder is $o_{\mathrm{P}}(N^{-1/2})$ as in Lemma C.9(ii).

Step 3 (Pairing side of (C.5)): Decompose $P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}_o'\cdot) = \phi_{\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o} + \widehat{\Delta P}(\boldsymbol{\gamma}_o'\cdot) + \hat{q}$ as in Step 5 of Lemma C.12, with $\phi_{\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o}(w) = DP_o(\boldsymbol{\gamma}_o'w)(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o)'w$. By Lemma C.14(iii) and the bound of Step 2, $\|\hat{d}_\Lambda - h_\Lambda\|_\infty \le C(K_N^{-(1+\alpha)} + \rho_N)$.

(a) $\langle \phi_{\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o}, \hat{d}_\Lambda\rangle_{P_o} = \langle \phi_{\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o}, h_\Lambda\rangle_{P_o} + O(|||\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o|||(K_N^{-(1+\alpha)}+\rho_N)) = \langle \Lambda^{\mathrm{lf}}, \hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o\rangle + o_{\mathrm{P}}(1)|||\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o|||$, by $\|\phi_U\|_{P_o} \le C|||U|||$ and Lemma C.14(ii).

(b) $\widehat{\Delta P} = \nabla(b'_{K_N}\hat{\xi} - G_o)$ on $\operatorname{Int}\mathcal{Z}$ lies in $\mathcal{T}$, and $\|\widehat{\Delta P}\|_{P_o} \le C\rho_N$, since $P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'w) - P_o(\boldsymbol{\gamma}_o'w) - \widehat{\Delta P}(\boldsymbol{\gamma}_o'w) = P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'w) - P_{\hat{\xi}}(\boldsymbol{\gamma}_o'w)$, which is bounded by $2MR_W|||\hat{\boldsymbol{\gamma}}-\boldsymbol{\gamma}_o|||$ by Lemma B.3(i). Hence $\langle \widehat{\Delta P}, \hat{d}_\Lambda\rangle_{P_o} = \langle \widehat{\Delta P}, h_\Lambda\rangle_{P_o} + O(\rho_N(K_N^{-(1+\alpha)} + \rho_N)) = \int_{\mathcal{Z}} \Lambda_P(z)'\widehat{\Delta P}(z)dz + o_{\mathrm{P}}(N^{-1/2})$, by Lemma C.14(i), $\sqrt{N}\rho_N K_N^{-(1+\alpha)} \to 0$, and $\sqrt{N}\rho_N^2 \to 0$.

(c) $|\langle \hat{q}, \hat{d}_\Lambda\rangle_{P_o}| \le C|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||(K_N\rho_N + h_N) = o_{\mathrm{P}}(1)|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||$ by Lemma C.5(ii), as in Step 5(c) of Lemma C.12.

Substituting Steps 2 and 3 into (C.5) gives the display. □

Many functionals of $\boldsymbol{\gamma}$ are regular with $\Lambda_P = 0$ and are covered by Theorem 3. The following lemma provides the primary example with a nonzero propensity derivative, an average of the propensity against a weight on the covariates and the judge. Regularity rests on the smoothness of the weight once it is carried to the index. For a set A whose boundary meets the interior of $\mathcal{Z}$ and a nonzero $c \in \mathbb{R}^2$, the indicator weight $\Upsilon(W) = \mathbb{1}\{Z \in A\}c$ fails the condition on $z \mapsto p_Z(z)\mathbb{E}[\Upsilon(W)\,|\,Z = z]$ below. For this weight $p_Z(z)\mathbb{E}[\Upsilon(W)\,|\,Z = z] = p_Z(z)\mathbb{1}\{z \in A\}c$, which jumps across ∂A.

Lemma C.16 (weighted averages of the propensity)**.** *Let $\Upsilon : \operatorname{supp}(W) \to \mathbb{R}^2$ be bounded and measurable, with $z \mapsto p_Z(z)\mathbb{E}[\Upsilon(W)\,|\,Z = z]$ of class $C^{1,\alpha}(\mathcal{Z})$. Under Assumptions 4 and 5, the functional $\Lambda(\theta) := \mathbb{E}[\Upsilon(W)'P(\boldsymbol{\gamma}'W)]$ is regular, with $\Lambda_{\boldsymbol{\gamma}} = \mathbb{E}[W\Upsilon(W)'DP_o(Z)]$ and $\Lambda_P(z) = p_Z(z)\mathbb{E}[\Upsilon(W)\,|\,Z = z]$.*

Proof. Fix $C > 0$ and $\theta = (\boldsymbol{\gamma}, P_\xi)$ with $\xi \in \Xi(K_N)$, $|||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o||| \le C\rho_N$, and $\mathsf{L}(\theta, \theta_o) \le C\rho_N$, and write $\boldsymbol{\gamma}_u := \boldsymbol{\gamma}_o + u(\boldsymbol{\gamma} - \boldsymbol{\gamma}_o)$ for $u \in [0, 1]$. Conditioning on Z, $\mathbb{E}[\Upsilon(W)'(P_\xi - P_o)(Z)] = \int_{\mathcal{Z}} \Lambda_P(z)'(P_\xi - P_o)(z)dz$. For the index part, the segment $\{\boldsymbol{\gamma}_u'w\}$ lies in the knot box, so

$$P_\xi(\boldsymbol{\gamma}'w) - P_\xi(\boldsymbol{\gamma}_o'w) = \left(\int_0^1 DP_\xi(\boldsymbol{\gamma}_u'w)du\right)(\boldsymbol{\gamma} - \boldsymbol{\gamma}_o)'w,$$

and $\Upsilon(W)'DP_o(Z)(\boldsymbol{\gamma}-\boldsymbol{\gamma}_o)'W = \langle W\Upsilon(W)'DP_o(Z), \boldsymbol{\gamma}-\boldsymbol{\gamma}_o\rangle$ by the symmetry of DP_o, so

$$R_\Lambda(\theta) = \mathbb{E}\left[\Upsilon(W)'\left(\int_0^1 \left(DP_\xi(\boldsymbol{\gamma}_u'W) - DP_o(\boldsymbol{\gamma}_o'W)\right)du\right)(\boldsymbol{\gamma}-\boldsymbol{\gamma}_o)'W\right].$$

Each $(\boldsymbol{\gamma}_u, P_\xi)$ lies in $\mathcal{U}_N$ with C enlarged, since $\mathsf{L}((\boldsymbol{\gamma}_u, P_\xi), \theta_o) \le \mathsf{L}(\theta, \theta_o) + 2MR_W|||\boldsymbol{\gamma}_u - \boldsymbol{\gamma}|||$ by Lemma B.3(i), so Lemma C.5(ii) gives $|R_\Lambda(\theta)| \le \|\Upsilon\|_\infty R_W|||\boldsymbol{\gamma}-\boldsymbol{\gamma}_o||| \cdot C(K_N\rho_N + h_N)$, uniformly over the set, and $K_N\rho_N + h_N \to 0$. Condition (i) of Definition 3 follows from the same computation along the path $(\boldsymbol{\gamma}_t, P_o + t\nabla\delta G)$, whose derivative at $t=0$ is

$$\mathbb{E}\left[\Upsilon(W)'DP_o(Z)V'W\right] + \mathbb{E}\left[\Upsilon(W)'\nabla\delta G(Z)\right] = \langle\Lambda_{\boldsymbol{\gamma}}, V\rangle + \int_{\mathcal{Z}} \Lambda_P(z)'\nabla\delta G(z)dz$$

by dominated convergence and the two identities above, since P_o and δG are continuously differentiable on the knot box and the derivative of $\boldsymbol{\gamma}_t$ at $t=0$ is V. □

Proof of Theorem C.1. With probability tending to one $\hat\theta \in \mathcal{U}_N$, so Definition 3(ii) gives $\Lambda(\hat\theta) - \Lambda(\theta_o) = \langle\Lambda_{\boldsymbol{\gamma}}, \hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\rangle + \int_{\mathcal{Z}} \Lambda_P(z)'\widehat{\Delta P}(z)dz + o_{\mathrm{P}}(N^{-1/2} + |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||)$, where $\widehat{\Delta P}(z) = P_{\hat\xi}(z) - P_o(z)$ for $z \in \mathcal{Z}$. Lemma C.15 replaces the integral, giving

$$\Lambda(\hat\theta) - \Lambda(\theta_o) = \langle\Lambda_{\boldsymbol{\gamma}} - \Lambda^{\mathrm{lf}}, \hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\rangle + (\mathbb{P}_N - \mathbb{P})\ell[\nabla(b_{K_N}'a_N(\Lambda))(\boldsymbol{\gamma}_o'\cdot)] + o_{\mathrm{P}}(N^{-1/2} + |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||).$$

By Theorem 3, $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| = O_{\mathrm{P}}(N^{-1/2})$ and $\mathrm{vec}(\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o) = (I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+\mathbb{P}_N s^{\mathrm{eff}}_{\boldsymbol{\gamma}} + o_{\mathrm{P}}(N^{-1/2})$, so the first term equals $\mathbb{P}_N[v_\Lambda' s^{\mathrm{eff}}_{\boldsymbol{\gamma}}] + o_{\mathrm{P}}(N^{-1/2}) = \mathbb{P}_N\ell[\phi^{\mathrm{eff}}_{v_\Lambda}] + o_{\mathrm{P}}(N^{-1/2})$, using $v's^{\mathrm{eff}}_{\boldsymbol{\gamma}} = \ell[\phi^{\mathrm{eff}}_v]$ from Lemma C.13. For the second term, $\mathbb{E}\ell[\nabla(b_{K_N}'a_N(\Lambda))(\boldsymbol{\gamma}_o'\cdot)] = 0$ by Lemma C.8 at θ_o, and $\mathbb{P}_N\ell[\nabla(b_{K_N}'a_N(\Lambda))(\boldsymbol{\gamma}_o'\cdot) - h_\Lambda]$ has mean zero and variance $\|\nabla(b_{K_N}'a_N(\Lambda)) - h_\Lambda\|^2_{P_o}/N \le CK_N^{-2(1+\alpha)}/N$ by the second-moment identity of Lemma C.13 and Lemma C.14(iii), hence is $o_{\mathrm{P}}(N^{-1/2})$. Put together, $\sqrt{N}(\Lambda(\hat\theta) - \Lambda(\theta_o)) = \sqrt{N}\mathbb{P}_N\ell[r_\Lambda] + o_{\mathrm{P}}(1)$ with $r_\Lambda = h_\Lambda + \phi^{\mathrm{eff}}_{v_\Lambda}$, which is the representation with $\tilde\psi_\Lambda = \ell[r_\Lambda]$ by Lemma C.14(iv). The summands are i.i.d. with mean zero and are bounded, since h_Λ and $\phi^{\mathrm{eff}}_{v_\Lambda}$ are bounded and $P_o^t \ge \kappa$, so the central limit theorem gives the normal limit with variance $\sigma^2_\Lambda = \mathbb{E}[\tilde\psi^2_\Lambda]$. A linear combination of finitely many regular functionals is regular, with $\Lambda_{\boldsymbol{\gamma}}$ and Λ_P combined linearly, so the representation holds jointly and joint normality follows from the Cramér–Wold device. For the efficiency claim, Definition 3(i) states that Λ is differentiable at θ_o along the submodels $t \mapsto (\boldsymbol{\gamma}_t, P_o + t\nabla\delta G)$, which are differentiable in quadratic mean with scores $v's_{\boldsymbol{\gamma}} + s_P[\nabla\delta G]$ spanning a dense subset of $\mathcal{T}_{\theta_o}$, with derivative $\dot\Lambda$, and Lemma C.14(iv) identifies $\tilde\psi_\Lambda$ as the Riesz representer of $\dot\Lambda$ on $\mathcal{T}_{\theta_o}$. The convolution theorem (Van der Vaart,

1998, Theorem 25.20) then bounds the asymptotic variance of every regular estimator of $\Lambda(\theta_o)$ below by $\mathbb{E}[\tilde{\psi}_\Lambda^2]$. □

C.2 Variance Estimation

Theorem C.1 gives the asymptotic variance σ_Λ^2 of a regular functional as the second moment of its efficient influence function. This subsection shows that the plug-in of Remark 3, with the spectral cut-off of Section 4.2, recovers it, and that the t and Wald statistics built from it are asymptotically standard. The estimator of the index coefficients is the case $\Lambda_P = 0$.

Notation. A sieve direction is a perturbation of the first-stage parameter $\theta = (\boldsymbol{\gamma}, \xi)$ at θ_o: a pair (V, a) with $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ moving the index coefficients and $a \in \mathbb{R}^{d_{K_N}}$ moving the surplus coefficients ξ. Its score is the derivative at $\vartheta = 0$ of the log-likelihood along $\vartheta \mapsto \theta_o + \vartheta(V, a)$. That score and its estimate are

$$s_N[V,a] := \ell[\phi_V + \nabla(b'_{K_N}a)(\boldsymbol{\gamma}_o'\cdot)], \qquad \hat{s}_N[V,a] := \mathrm{vec}(V)'\Pi_{\hat{\boldsymbol{\gamma}}}\hat{s}_{\boldsymbol{\gamma}} + \hat{s}_\xi[a],$$

with $\hat{s}_{\boldsymbol{\gamma}}$ and $\hat{s}_\xi$ as in Remark 3, and the second-moment identity in the proof of Lemma C.13 gives $\mathbb{E}[s_N[V,a]^2] = \|\phi_V + \nabla(b'_{K_N}a)\|_{P_o}^2$. Stacking $(\mathrm{vec}(V), a)$ identifies these pairs with a Euclidean space of dimension $\dim(\mathcal{T}_{\boldsymbol{\gamma}_o}) + d_{K_N}$, in which the two second moments are the matrices $I_N := \mathbb{E}[s_N s_N']$ and $\hat{I}_N := \mathbb{P}_N \hat{s}_N \hat{s}_N'$, the derivative of a regular functional Λ on the sieve directions is the vector $\dot{\Lambda}_N$ with

$$\dot{\Lambda}_N[V,a] := \dot{\Lambda}(s_N[V,a]) = \langle \Lambda_{\boldsymbol{\gamma}}, V\rangle + \int_{\mathcal{Z}} \Lambda_P(z)'\nabla(b'_{K_N}a)(z)dz,$$

and $|(V,a)|$ is the Euclidean norm of the stack. Let $\hat{\dot{\Lambda}}_N$ be the same expression with $(\hat{\Lambda}_{\boldsymbol{\gamma}}, \hat{\Lambda}_P)$ in place of $(\Lambda_{\boldsymbol{\gamma}}, \Lambda_P)$. Fix a threshold $\mathsf{tol}_N > 0$, let $\hat{I}_{N,\xi\xi} = \sum_m \hat{\lambda}_m \hat{p}_m \hat{p}_m'$ be a spectral decomposition of the surplus block of $\hat{I}_N$, and let $\hat{R}_N$ be the span of the index directions $(\mathrm{vec}(V), 0)$ with $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$ and of the surplus directions $(0, \hat{p}_m)$ with $\hat{\lambda}_m \geq \mathsf{tol}_N$. The variance estimator is

$$\hat{\sigma}_\Lambda^2 := \sup\left\{ \frac{\hat{\dot{\Lambda}}_N[V,a]^2}{\mathbb{P}_N\left(\hat{s}_N[V,a]^2\right)} : (\mathrm{vec}(V),a) \in \hat{R}_N,\ \mathbb{P}_N\left(\hat{s}_N[V,a]^2\right) > 0 \right\}, \tag{C.6}$$

the expression in Remark 3 with the spectral cut-off of Section 4.2. The population

counterpart at the sieve is $\sigma^2_{\Lambda,N} := \sup\{\dot\Lambda_N[V,a]^2/\|\phi_V + \nabla(b'_{K_N}a)\|^2_{P_o}\}$ over the directions at which the denominator is positive.

Lemma C.17 (the sieve variance)**.** *Let Λ be a regular functional. Under Assumptions 1, 3, 4 and 5, there is C with*

$$\sigma^2_\Lambda - CK_N^{-(1+\alpha)} \le \sigma^2_{\Lambda,N} \le \sigma^2_\Lambda,$$

and $|\dot\Lambda_N[V,a]| \le \sigma_\Lambda\|\phi_V+\nabla(b'_{K_N}a)\|_{P_o}$ for every sieve direction. The direction (V_Λ, a^*), with V_Λ the matrix form of v_Λ and $a^* := a_N(\Lambda) - a_N(v_\Lambda)$, satisfies $\|\phi_{V_\Lambda} + \nabla(b'_{K_N}a^*)\|^2_{P_o} = \sigma^2_\Lambda + O(K_N^{-(1+\alpha)})$, $\dot\Lambda_N[V_\Lambda, a^*] = \sigma^2_\Lambda + O(K_N^{-(1+\alpha)})$, $\|\nabla(b'_{K_N}a^*)\|_{L^\infty(\mathcal{Z})} \le C$ and $|(V_\Lambda, a^*)| \le CK_N$.

Proof. For a sieve direction, the field $\nabla(b'_{K_N}a)$ is a smooth gradient field on $\bar{\mathcal{Z}}$ and hence lies in $\mathcal{T}$, so $s_N[V,a] \in \mathcal{T}_{\theta_o}$, and the definition of $\dot\Lambda$ gives the second expression for $\dot\Lambda_N[V,a]$. Lemma C.14(iv) makes $\dot\Lambda$ bounded on $\mathcal{T}_{\theta_o}$ with Riesz representer $\tilde\psi_\Lambda = \ell[r_\Lambda]$, so $\dot\Lambda_N[V,a] = \langle r_\Lambda, \phi_V + \nabla(b'_{K_N}a)\rangle_{P_o}$, which is the second display by the Cauchy–Schwarz inequality, and $\sigma^2_\Lambda = \|r_\Lambda\|^2_{P_o} = \sup\{\dot\Lambda(s)^2/\mathbb{E}[s^2] : s \in \mathcal{T}_{\theta_o},\ \mathbb{E}[s^2] > 0\}$. The sieve directions form a subspace of $\mathcal{T}_{\theta_o}$, which gives the upper bound on $\sigma^2_{\Lambda,N}$.

For the lower bound, take (V_Λ, a^*) as stated. Then $\phi_{V_\Lambda} + \nabla(b'_{K_N}a^*) = \phi^{\text{eff}}_{N,v_\Lambda} + \nabla(b'_{K_N}a_N(\Lambda))$, and $\|\phi^{\text{eff}}_{N,v_\Lambda} - \phi^{\text{eff}}_{v_\Lambda}\|_{P_o} \le CK_N^{-(1+\alpha)}$ by Lemma C.3(i) while $\|\nabla(b'_{K_N}a_N(\Lambda)) - h_\Lambda\|_{L^\infty(\mathcal{Z})} \le CK_N^{-(1+\alpha)}$ by Lemma C.14(iii), so that field differs from r_Λ by $CK_N^{-(1+\alpha)}$ in $\|\cdot\|_{P_o}$. The two displayed values follow, and their ratio is $\sigma^2_\Lambda + O(K_N^{-(1+\alpha)})$ unless $\sigma_\Lambda = 0$, in which case $\dot\Lambda = 0$ and there is nothing to prove. The sup-norm bound is Lemmas C.3(ii) and C.14(iii). The bound on the stack holds because $a_N(v_\Lambda)$ and $a_N(\Lambda)$ are quasi-interpolant coefficients of functions bounded in $C^{2,\alpha}(\bar{\mathcal{Z}})$, hence bounded one by one, and $d_{K_N} \asymp K_N^2$. □

Lemma C.18 (the sieve information)**.** *Under the conditions of Theorem C.1:*

(i) $\sup_\omega |s_N[V,a](\omega)| \le CK_N|(V,a)|$ *and* $\|\phi_V + \nabla(b'_{K_N}a)\|_{P_o} \le C|(V,a)|$ *for any sieve direction.*

(ii) $|||V|||^2 + \|\nabla(b'_{K_N}a)\|^2_{L^2(\mathcal{Z})} \le C\mathbb{E}[s_N[V,a]^2]$ *for any sieve direction.*

(iii) $|\mathbb{P}_N\hat s_N[V,a]^2 - \mathbb{E}[s_N[V,a]^2]| \le O_{\mathrm{P}}(K_N\rho_N)|(V,a)|^2 + o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2]$ *uniformly over sieve directions.*

(iv) If $K_N\rho_N = o(\mathsf{tol}_N)$, then with probability tending to one every direction with $(\mathrm{vec}(V), a) \in \hat{R}_N$ satisfies $\mathbb{E}[s_N[V,a]^2] \geq \min\{1, \mathsf{tol}_N\}|(V,a)|^2/C$, and $\mathbb{P}_N \hat{s}_N[V,a]^2 = (1 + o_{\mathrm{P}}(1))\mathbb{E}[s_N[V,a]^2]$ uniformly over these directions.

Proof. (i) At a point of the knot box at most q^2 of the fields ∇b_j are nonzero, and each is bounded by CK_N by the Markov inequality on its cell, so $|\nabla(b'_{K_N}a)(z)| \leq CK_N|a|$. The index part is bounded by $CR_W|||V|||$ and $P_o^t \geq \kappa$, which gives the first claim. For the second, $\|\phi_V + \nabla(b'_{K_N}a)\|_{P_o}^2 \leq C(|||V|||^2 + \|\nabla(b'_{K_N}a)\|_{L^2(\bar{\mathcal{Z}})}^2) \leq C|(V,a)|^2$, the last step because the Gram matrix of $\{\nabla b_j\}$ over the knot box has bounded operator norm on a quasi-uniform mesh.

(ii) The field $\nabla(b'_{K_N}a)$ is a gradient field on $\bar{\mathcal{Z}}$ and hence lies in $\mathcal{T}$, while ϕ_V^{eff} is the residual of ϕ_V on $\mathcal{T}$. Hence $\mathbb{E}[s_N[V,a]^2] = \|\phi_V + \nabla(b'_{K_N}a)\|_{P_o}^2 \geq \|\phi_V^{\mathrm{eff}}\|_{P_o}^2 = \mathrm{vec}(V)' I_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\mathrm{eff}} \mathrm{vec}(V) \geq \lambda_{\min}^{\mathrm{eff}}|||V|||^2$, where $\lambda_{\min}^{\mathrm{eff}}$ denotes the smallest eigenvalue of $I_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\mathrm{eff}}$ on $\mathcal{T}_{\boldsymbol{\gamma}_o}$, a space of fixed finite dimension on which Lemma C.1 makes that eigenvalue positive. For the field, $\|\nabla(b'_{K_N}a)\|_{P_o} \leq \mathbb{E}[s_N[V,a]^2]^{1/2} + \|\phi_V\|_{P_o}$ with $\|\phi_V\|_{P_o} \leq C|||V|||$, and the norms $\|\cdot\|_{P_o}$ and $\|\cdot\|_{L^2(\mathcal{Z})}$ are equivalent on $\mathcal{Z}$ by Assumptions 4(b) and (f).

(iii) Write $\mathbb{P}_N \hat{s}_N[V,a]^2 - \mathbb{E}[s_N[V,a]^2] = (\mathbb{P}_N s_N[V,a]^2 - \mathbb{E}[s_N[V,a]^2]) + \mathbb{P}_N(\hat{s}_N[V,a]^2 - s_N[V,a]^2)$. For the first term the summands are independent with $\|s_N\|^2 \leq CK_N^2$ by (i) and $\|\mathbb{E}[\|s_N\|^2 s_N s_N']\|_{\mathrm{op}} \leq CK_N^2$, so a matrix Bernstein inequality gives $\|\mathbb{P}_N s_N s_N' - I_N\|_{\mathrm{op}} = O_{\mathrm{P}}(K_N\sqrt{\log d_{K_N}/N} + K_N^2 \log d_{K_N}/N) = o_{\mathrm{P}}(K_N\rho_N)$ under Assumption 5(a). For the second term,

$$\left|\mathbb{P}_N(\hat{s}_N[V,a]^2 - s_N[V,a]^2)\right| \leq 2(\mathbb{P}_N s_N[V,a]^2)^{1/2}(\mathbb{P}_N \Delta[V,a]^2)^{1/2} + \mathbb{P}_N \Delta[V,a]^2$$

for $\Delta[V,a] := \hat{s}_N[V,a] - s_N[V,a]$. Since $\Pi_{\hat{\boldsymbol{\gamma}}}\mathrm{vec}(V) = \mathrm{vec}(V - \langle V, \hat{\boldsymbol{\gamma}}\rangle\hat{\boldsymbol{\gamma}})$ for $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, the difference splits as $\Delta[V,a] = \Delta_1 + \Delta_2 + \Delta_3$ with

$$\begin{aligned}
\Delta_1 &:= \sum_t T^t \left(DP_o(Z)V'W + \nabla(b'_{K_N}a)(Z)\right)^t \left(\frac{1}{P_{\hat{\xi}}^t(\hat{\boldsymbol{\gamma}}'W)} - \frac{1}{P_o^t(Z)}\right), \\
\Delta_2 &:= \sum_t \frac{T^t}{P_{\hat{\xi}}^t(\hat{\boldsymbol{\gamma}}'W)} \left(\nabla(b'_{K_N}a)(\hat{\boldsymbol{\gamma}}'W) - \nabla(b'_{K_N}a)(Z)\right)^t, \\
\Delta_3 &:= \sum_t \frac{T^t}{P_{\hat{\xi}}^t(\hat{\boldsymbol{\gamma}}'W)} \left(DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'W)(V - \langle V, \hat{\boldsymbol{\gamma}}\rangle\hat{\boldsymbol{\gamma}})'W - DP_o(Z)V'W\right)^t.
\end{aligned}$$

The term Δ_1 satisfies $|\Delta_1| \leq \kappa^{-2}|P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'W) - P_o(Z)||DP_o(Z)V'W + \nabla(b'_{K_N}a)(Z)|$. Here

the factor $|DP_o(Z)V'W + \nabla(b'_{K_N}a)(Z)|$ is at most $CK_N|(V,a)|$ by the proof of (i), and $\mathbb{P}_N|P_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'\cdot) - P_o(\boldsymbol{\gamma}'_o\cdot)|^2 = O_{\mathrm{P}}(\rho_N^2)$ by Lemma C.4 applied to the class $\{|P(\boldsymbol{\gamma}'\cdot) - P_o(\boldsymbol{\gamma}'_o\cdot)|^2 : (\boldsymbol{\gamma}, P) \in \mathcal{U}_N\}$. Hence $\mathbb{P}_N\Delta_1^2 = O_{\mathrm{P}}(K_N^2\rho_N^2)|(V,a)|^2$. The term Δ_2 satisfies

$$|\Delta_2| \le \kappa^{-1}R_W|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| \sup_{z\in[Z,\hat{\boldsymbol{\gamma}}'W]} \|D^2(b'_{K_N}a)(z)\|,$$

and $\rho_N = o(h_N)$ confines the segment $[Z, \hat{\boldsymbol{\gamma}}'W]$ to the cell of Z and its neighbours. The supremum of a polynomial over a cell is at most Ch_N^{-1} times its L^2 norm there, so the square of that supremum is at most $a'A(z)a$ for the matrix $A(z) := Ch_N^{-2}\int_{\mathcal{V}(z)}(D^2b)(D^2b)'$ over the neighbourhood $\mathcal{V}(z)$ of the cell of z, with $\|A\|_{\mathrm{op}} \le CK_N^4$ and $\|\mathbb{E}A\|_{\mathrm{op}} \le CK_N^2$ by the Markov inequality over full cells. The same Bernstein bound gives $\|\mathbb{P}_NA\|_{\mathrm{op}} = O_{\mathrm{P}}(K_N^2)$, hence $\mathbb{P}_N\Delta_2^2 = O_{\mathrm{P}}(K_N^2\rho_N^2)|(V,a)|^2$. The term Δ_3 does not involve a. Since $\langle V, \boldsymbol{\gamma}_o\rangle = 0$ for $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, the term in parentheses in Δ_3 equals $(DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'W) - DP_o(Z))V'W - \langle V, \hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\rangle DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'W)\hat{\boldsymbol{\gamma}}'W$, and

$$|\Delta_3| \le C|||V||| \left(\|DP_{\hat{\xi}}(\hat{\boldsymbol{\gamma}}'W) - DP_o(Z)\| + |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||\right).$$

On the event $\hat{\theta} \in \mathcal{U}_N$, the first norm is at most $CK_N^2\rho_N$ on $\mathcal{B}_N^c$ by Lemma C.5(i), as in Step 3 of the proof of Lemma C.9, and at most C on $\mathcal{B}_N$ by the constraints of $\Xi(K_N)$. The set $\mathcal{B}_N$ does not depend on the sample and has probability $O(h_N)$, which gives $\mathbb{P}_N\mathbb{1}_{\mathcal{B}_N} = O_{\mathrm{P}}(h_N)$. Hence $\mathbb{P}_N\Delta_3^2 \le C|||V|||^2(K_N^4\rho_N^2 + \mathbb{P}_N\mathbb{1}_{\mathcal{B}_N} + |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o|||^2) = o_{\mathrm{P}}(1)|||V|||^2$. By the bound on the first term, the bounds on Δ_1, Δ_2 and Δ_3, and (ii),

$$(\mathbb{P}_Ns_N[V,a]^2)^{1/2} \le \mathbb{E}[s_N[V,a]^2]^{1/2} + o_{\mathrm{P}}((K_N\rho_N)^{1/2})|(V,a)|,$$
$$(\mathbb{P}_N\Delta[V,a]^2)^{1/2} \le O_{\mathrm{P}}(K_N\rho_N)|(V,a)| + o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2]^{1/2}.$$

Inserting both into the display for the second term bounds it by $O_{\mathrm{P}}(K_N\rho_N)|(V,a)|^2 + o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2]$, using $\mathbb{E}[s_N[V,a]^2] \le C|(V,a)|^2$ from (i) and the inequality of arithmetic and geometric means for the product of $(K_N\rho_N)^{1/2}|(V,a)|$ and $\mathbb{E}[s_N[V,a]^2]^{1/2}$. With the bound on the first term, this is (iii).

(iv) Let $(\mathrm{vec}(V), a) \in \hat{R}_N$. Then a lies in the span of the eigenvectors $\hat{p}_m$ with $\hat{\lambda}_m \ge \mathsf{tol}_N$, and $\mathbb{P}_N\hat{s}_N[0,a]^2 = a'\hat{I}_{N,\xi\xi}a \ge \mathsf{tol}_N|a|^2$. By (iii) with $V = 0$ and $K_N\rho_N = o(\mathsf{tol}_N)$, $\mathbb{E}[s_N[0,a]^2] \ge (1 - o_{\mathrm{P}}(1))\mathsf{tol}_N|a|^2$ uniformly in such a. By (ii) and the equivalence of $\|\cdot\|_{P_o}$ and $\|\cdot\|_{L^2(\mathcal{Z})}$, $\mathbb{E}[s_N[V,a]^2] \ge (|||V|||^2 + \mathbb{E}[s_N[0,a]^2])/C$, which gives the first claim. On these directions $O_{\mathrm{P}}(K_N\rho_N)|(V,a)|^2 = o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2]$, because $K_N\rho_N \to 0$ and $K_N\rho_N = o(\mathsf{tol}_N)$, and (iii) gives the second claim. □

Proposition C.1 (consistency of the variance estimator)**.** *Let $\Lambda_1, \dots, \Lambda_r$ be regular functionals whose estimated derivatives satisfy $|||\hat{\Lambda}_{\boldsymbol{\gamma},j} - \Lambda_{\boldsymbol{\gamma},j}||| + \|\hat{\Lambda}_{P,j} - \Lambda_{P,j}\|_{L^2(\mathcal{Z})} = o_{\mathrm{P}}(1)$, let $\hat{\Sigma}$ be the symmetric matrix with $t'\hat{\Sigma}t$ equal to (C.6) for the functional $\sum_j t_j\Lambda_j$ and the derivative estimate $\sum_j t_j\hat{\dot{\Lambda}}_{N,j}$, and let Σ be the asymptotic covariance matrix of $(\Lambda_j(\hat{\theta}))_{j=1}^r$ in Theorem C.1, assumed nonsingular. Suppose the threshold satisfies*

$$K_N\rho_N = o(\mathsf{tol}_N) \qquad \text{and} \qquad \mathsf{tol}_N = o(K_N^{-2}), \tag{C.7}$$

which is possible under Assumption 5(a). Then, under the conditions of Theorem C.1, $\hat{\Sigma} \to_p \Sigma$, the Wald statistic

$$\mathrm{W} = N(\Lambda(\hat{\theta}) - \Lambda(\theta_o))'\hat{\Sigma}^{-1}(\Lambda(\hat{\theta}) - \Lambda(\theta_o)) \tag{C.8}$$

converges in distribution to χ^2_r, and for $r = 1$ the t-statistic $\sqrt{N}(\Lambda(\hat{\theta}) - \Lambda(\theta_o))/\hat{\sigma}_\Lambda$ converges to a standard normal.

Proof. On the event that $\mathbb{P}_N\hat{s}_N[V,a]^2 > 0$ for every nonzero direction stacking into $\hat{R}_N$, the Cauchy–Schwarz inequality for the inner product that $\hat{I}_N$ defines on $\hat{R}_N$ makes (C.6) a quadratic form in $\hat{\dot{\Lambda}}_N$, and $\hat{\Sigma}$ is well defined. A linear combination $\sum_j t_j\Lambda_j$ is regular with derivative $\sum_j t_j\dot{\Lambda}_{N,j}$, and $t'\hat{\Sigma}t$ is (C.6) for that combination, so it suffices to treat a single Λ and to conclude by the Cramér–Wold device, the continuity of the inverse at the nonsingular limit, and Slutsky.

A threshold exists: $\mathsf{tol}_N := K_N^{-2}(K_N^3\rho_N)^{1/2}$ satisfies (C.7) because $K_N^3\rho_N \to 0$ under Assumption 5(a). By Lemma C.18(iv) and (C.7), the event of the first paragraph has probability tending to one, and

$$\left|\mathbb{P}_N\hat{s}_N[V,a]^2 - \mathbb{E}[s_N[V,a]^2]\right| = o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2] \tag{C.9}$$

uniformly over the directions stacking into $\hat{R}_N$. For a in the span of the eigenvectors $\hat{p}_m$ with $\hat{\lambda}_m < \mathsf{tol}_N$, $\mathbb{P}_N\hat{s}_N[0,a]^2 \le \mathsf{tol}_N|a|^2$, and Lemma C.18(iii) gives $\mathbb{E}[s_N[0,a]^2] \le (1 + o_{\mathrm{P}}(1))\mathsf{tol}_N|a|^2$ uniformly in such a.

For the numerator,

$$|(\hat{\dot{\Lambda}}_N - \dot{\Lambda}_N)[V,a]| \le |||\hat{\Lambda}_{\boldsymbol{\gamma}} - \Lambda_{\boldsymbol{\gamma}}|||\,|||V||| + \|\hat{\Lambda}_P - \Lambda_P\|_{L^2(\mathcal{Z})}\|\nabla(b'_{K_N}a)\|_{L^2(\mathcal{Z})},$$

and both $|||V|||$ and $\|\nabla(b'_{K_N}a)\|_{L^2(\mathcal{Z})}$ are bounded by $C\mathbb{E}[s_N[V,a]^2]^{1/2}$ by Lemma C.18(ii).

Hence $|(\hat{\dot{\Lambda}}_N - \dot{\Lambda}_N)[V,a]| \le o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2]^{1/2}$ uniformly.

For the upper bound, (C.9) and the previous display give

$$\hat{\sigma}_\Lambda^2 \le (1+o_{\mathrm{P}}(1)) \sup \frac{\left(|\dot{\Lambda}_N[V,a]| + o_{\mathrm{P}}(1)\mathbb{E}[s_N[V,a]^2]^{1/2}\right)^2}{\mathbb{E}[s_N[V,a]^2]}$$
$$\le (1+o_{\mathrm{P}}(1))\left(\sigma_{\Lambda,N} + o_{\mathrm{P}}(1)\right)^2 \le \sigma_\Lambda^2 + o_{\mathrm{P}}(1)$$

by Lemma C.17.

For the lower bound, write $a^* = a_R + a_D$ for the direction (V_Λ, a^*) of Lemma C.17, with a_R in the span of the eigenvectors $\hat{p}_m$ with $\hat{\lambda}_m \ge \mathsf{tol}_N$ and a_D in the span of the others. The direction (V_Λ, a_R) then stacks into $\hat{R}_N$. Since $|a_D| \le |(V_\Lambda, a^*)| \le CK_N$ by Lemma C.17, the second paragraph gives $\mathbb{E}[s_N[0,a_D]^2] \le (1+o_{\mathrm{P}}(1))C\mathsf{tol}_N K_N^2 = o_{\mathrm{P}}(1)$ by (C.7). The score is linear in the direction, with $s_N[V_\Lambda, a_R] = s_N[V_\Lambda, a^*] - s_N[0,a_D]$, and Lemma C.17 and the triangle inequality in $L^2(\mathbb{P})$ give $\mathbb{E}[s_N[V_\Lambda,a_R]^2] = \sigma_\Lambda^2 + o_{\mathrm{P}}(1)$. Likewise $\dot{\Lambda}_N[V_\Lambda,a_R] = \dot{\Lambda}_N[V_\Lambda,a^*] - \dot{\Lambda}_N[0,a_D] = \sigma_\Lambda^2 + o_{\mathrm{P}}(1)$, since $|\dot{\Lambda}_N[0,a_D]| \le \sigma_\Lambda \mathbb{E}[s_N[0,a_D]^2]^{1/2}$ by Lemma C.17. By (C.9) and the numerator bound, the ratio in (C.6) at (V_Λ, a_R) is $\sigma_\Lambda^2 + o_{\mathrm{P}}(1)$, which gives $\hat{\sigma}_\Lambda^2 \ge \sigma_\Lambda^2 + o_{\mathrm{P}}(1)$.

Theorem C.1 gives $\sqrt{N}(\Lambda(\hat{\theta}) - \Lambda(\theta_o)) \to_d \mathcal{N}(0,\Sigma)$ jointly, and the limit laws follow. □

Remark 1 (the index coefficients). For $\Lambda(\theta) = \langle A, \boldsymbol{\gamma}\rangle$ with $A \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, Definition 3 holds with $\Lambda_{\boldsymbol{\gamma}} = A$ and $\Lambda_P = 0$, so $f_\Lambda = 0$, $h_\Lambda = 0$, $\Lambda^{\mathrm{lf}} = 0$ and $\sigma_\Lambda^2 = \mathrm{vec}(A)'(I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+\mathrm{vec}(A)$ by Lemma C.14(iv). The derivative is known rather than estimated, and Proposition C.1 reads $\mathrm{vec}(A)'\hat{\Sigma}\mathrm{vec}(A) \to_p \mathrm{vec}(A)'(I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+\mathrm{vec}(A)$ for every $A \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, which is the consistency on $\mathcal{T}_{\boldsymbol{\gamma}_o}$ of the estimator of Remark 3 with the spectral cut-off of Section 4.2.

D Second-Stage Rates with Generated Regressors

The second stage uses the first-stage estimates to construct the regressors, and the regressors' uniform accuracy governs the sup-norm convergence of the effect functions. With W_i bounded, the index proxy obeys $\max_{1\le i\le N}\|\hat{Z}_i - Z_i\| \le |||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| \max_{1\le i\le N}\|W_i\| = O_{\mathrm{P}}(\rho_N)$ by Theorem 2(ii), and for sample points with $Z_i \in \mathcal{Z}_N$ the propensity proxy adds a Lipschitz index term to the interior propensity error,

$$|\hat{P}_i^t - P_o^t(Z_i)| \le M\|\hat{Z}_i - Z_i\| + \|P^t_{\hat{\xi}} - P_o^t\|_{L^\infty(\mathcal{Z}_N)} = O_{\mathrm{P}}(K_N\rho_N),$$

the propensity error dominating the $O_{\mathrm{P}}(\rho_N)$ index term.

Denote the second-stage regressor by $S_i = S_i^{t,r} = (P_i^t, Z_i^{t,r})$, as in (15), so that (14) reads $Y_i = X_i'\beta_{ts}(S_i) + \varepsilon_i$; we suppress the pair superscript where no confusion arises. Let $\hat{S}_i = (\hat{P}_i^t, \hat{Z}_i^{t,r})$ be the constructed regressor. The map $z \mapsto (P_o^t(z), z^{t,r})$ is injective under the full support condition on f, so S_i is a one-to-one transformation of Z_i, and conditioning on (S_i, X_i) is the same as conditioning on (Z_i, X_i). Let $\mathcal{S}^- \subseteq \mathcal{S}$ be compact subsets of the interior of $\mathrm{supp}(S_i)$ with $\mathcal{S}^-$ contained in the interior of $\mathcal{S}$. The uniform rates below are stated over $\mathcal{S}^-$, whereas estimation is based on the full t-subsample. The same rates hold when estimation is restricted to the cases with $\hat{S}_i \in \mathcal{S}$. The constructed regressor satisfies $\max_{i:S_i\in\mathcal{S}} \|\hat{S}_i - S_i\| = O_{\mathrm{P}}(K_N\rho_N)$, with the propensity coordinate binding, and Assumption D.1(e) below maintains this input as a generic rate ν_N.

Assumption D.1. *The following hold.*

(a) (Error moment and design.) $\mathbb{E}[\varepsilon_i^2 \,|\, T_i, Z_i, X_i] \le \bar{\sigma}^2$ *and* $\mathbb{E}[X_iX_i' \,|\, S_i] \ge \underline{c}\mathbf{I}_{d_X}$ *almost surely on* $\{S_i \in \mathcal{S}\}$ *for some constants* $\bar{\sigma}^2, \underline{c} > 0$.

(b) (Smoothness.) $\beta_{ts}(\cdot) \in C^\chi(\mathrm{supp}(S_i))$ *componentwise for some* $\chi \ge 2$*, and the second-stage spline order satisfies* $\tilde{q} \ge \chi + 1$.

(c) (Regressor density and mesh.) S_i *admits a density on* $\mathcal{S}$ *bounded above and below by positive constants, and the second-stage knot sequences are quasi-uniform.*

(d) (Sieve growth and coupling.) $\tilde{K}_N \to \infty$ *and* $\tilde{K}_N\nu_N \to 0$*, with* ν_N *the first-stage rate of Assumption D.1(e), and the threshold* tol_N *of the first-stage variance estimate in Section 4.2 satisfies* $K_N\rho_N = o(\mathsf{tol}_N)$.

(e) (First-stage passthrough.) *There is a sequence* $\nu_N \to 0$ *such that the first-stage estimators satisfy* $\|\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\|\| = O_{\mathrm{P}}(\nu_N)$ *and* $\|P_{\hat{\xi}}^t - P_o^t\|_{L^\infty(\mathcal{Z}_N)} = O_{\mathrm{P}}(\nu_N)$ *for* $t \in \{1, 2\}$*, on the interior regions* $\mathcal{Z}_N$ *of Theorem 2(ii).*

Assumption D.1(a) requires the error variance to be bounded and the case characteristics to have full rank conditional on almost every value of S_i in $\mathcal{S}$. The error bound conditions on the index Z_i, which is the same as conditioning on the regressor S_i. The full-rank condition is needed to identify $\beta_{ts}(\cdot)$ at each point of $\mathcal{S}$ and keeps the least-squares problem uniformly well-posed there. Assumption D.1(b) is used to control the spline bias. With $\tilde{q} \ge \chi + 1$, the same fit approximates $\beta_{ts}(\cdot)$ to $\tilde{K}_N^{-\chi}$ and its derivative $\partial_1\beta_{ts}(\cdot)$ in propensity to $\tilde{K}_N^{-(\chi-1)}$ in the sup norm. Assumption D.1(c) is needed to establish the uniform rate

on $\mathcal{S}^-$, since the positive density and the quasi-uniform knots keep the Gram matrix well-behaved there. Assumption D.1(d) caps the sieve dimension so that the variance vanishes, and couples the two sieves through $\tilde{K}_N \nu_N \to 0$, which makes the bias induced by the generated regressor vanish. It also requires the threshold tol_N to dominate the sampling error of the first-stage sample information, so that the directions the variance estimate retains are those the population information supports. Assumption D.1(e) allows a generic first-stage rate ν_N, which Theorem 2(ii) pins down to $\nu_N = K_N \rho_N$. $K_N \rho_N \to 0$ holds under $K_N^4 \log N = o(N)$. The uniform consistency rates stated in Theorem D.1 hold for any first-stage estimator attaining such a rate, irrespective of semiparametric efficiency.

The following theorem gives the rates of the second-stage estimators $\hat{\beta}_{ts}(\cdot)$, $\widehat{\partial_1 \beta}_{ts}(\cdot)$, $\hat{\mu}_{ts}(\cdot)$ and $\hat{\tau}_{ts}(\cdot)$ of Section 4.2, all based on $\hat{\delta}$. The first-stage error enters through the generated regressor $\hat{S}_i$.

Theorem D.1. *(Convergence rates of the second-stage estimator.) Suppose Assumptions 1, 3, 4, 5 and D.1 hold. Then, for each margin (t, s) and each component $\iota = 1, \ldots, d_X$,*

$$\|\hat{\beta}_{ts,\iota} - \beta_{ts,\iota}\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}\Big(\tilde{K}_N \sqrt{\log N / N} + \tilde{K}_N^{-\chi} + \nu_N\Big),$$
$$\|\widehat{\partial_1 \beta}_{ts,\iota} - \partial_1 \beta_{ts,\iota}\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}\Big(\tilde{K}_N^2 \sqrt{\log N / N} + \tilde{K}_N^{-(\chi-1)} + \tilde{K}_N \nu_N\Big).$$

Consequently, the margin-specific mean potential outcome $\hat{\mu}_{ts}$ and treatment effect $\hat{\tau}_{ts}$ converge uniformly at the derivative rate,

$$\sup_{(z,x)} |\hat{\mu}_{ts}(z,x) - \mu_{ts}(z,x)| = O_{\mathrm{P}}\Big(\tilde{K}_N^2 \sqrt{\log N / N} + \tilde{K}_N^{-(\chi-1)} + \tilde{K}_N \nu_N\Big),$$
$$\sup_{(z,x)} |\hat{\tau}_{ts}(z,x) - \tau_{ts}(z,x)| = O_{\mathrm{P}}\Big(\tilde{K}_N^2 \sqrt{\log N / N} + \tilde{K}_N^{-(\chi-1)} + \tilde{K}_N \nu_N\Big),$$

with the suprema running over (z, x) whose arguments $(P_o^t(z), z^{t,r})$ and $(P_o^s(z), z^{s,r})$ lie in $\mathcal{S}^-$.

Each rate splits into a sieve variance term, a sieve bias term, and a first-stage passthrough term, and the derivative carries one extra factor $\tilde{K}_N$ in each, the cost of differentiating the spline fit. The passthrough enters through the generated regressor $\hat{S}_i$, whose uniform error on $\mathcal{S}$ is $O_{\mathrm{P}}(\nu_N)$, and by the Lipschitz continuity of $\beta_{ts}(\cdot)$ it contributes ν_N to the level and $\tilde{K}_N \nu_N$ to the derivative, both vanishing under Assumption D.1(d). The correction is of second order in the first-stage error and is absorbed in the passthrough

term. The estimated effect function $\hat{\tau}_{ts}(\cdot)$ depends on $\widehat{\partial_1\beta}_{ts}(\cdot)$ through (8) and therefore converges at the derivative rate.

We now prove Theorem D.1. Fix the pair (t, s) and the remaining treatment r. All objects refer to the t-subsample. Write $S_i = (P_i^t, Z_i^{t,r})$, $\hat{S}_i = (\hat{P}_i^t, \hat{Z}_i^{t,r})$, and $\tilde{d} = (\tilde{K}_N+\tilde{q}-1)^2$. Let $\tilde{b}(\cdot) = \tilde{b}_{\tilde{K}_N}(\cdot) \in \mathbb{R}^{\tilde{d}}$ be the order-$\tilde{q}$ tensor B-spline basis on the quasi-uniform mesh of width $\tilde{h} \asymp \tilde{K}_N^{-1}$ covering $\mathrm{supp}(S_i)$. The regression basis is $\varphi_i := X_i \otimes \tilde{b}(S_i)$ and its generated version $\hat{\varphi}_i := X_i \otimes \tilde{b}(\hat{S}_i)$. The least squares problem of Section 4.2 has solution $\hat{\delta}^{\mathrm{ls}} = (\hat{\delta}_\iota^{\mathrm{ls}})_\iota$, the correction (19) maps it to $\hat{\delta} = (\hat{\delta}_\iota)_\iota$, and the fitted coefficients are $\hat{\beta}_{ts,\iota}(\cdot) = \tilde{b}(\cdot)'\hat{\delta}_\iota$. Throughout, all uniform statements about fitted functions are on $\mathcal{S}^-$, controlled by basis functions whose supports meet $\mathcal{S}^-$ and hence lie in $\mathcal{S}$ for $\tilde{K}_N$ large. Only observations whose regressors activate the retained basis functions enter the estimates below, and for N large these have $S_i \in \mathcal{S}$, where the interior passthrough of Assumption D.1(e) applies. By that condition and boundedness,

$$\Delta_N := \max_{i \le N:\, S_i \in \mathcal{S}} \|\hat{S}_i - S_i\| = O_{\mathrm{P}}(\nu_N). \tag{D.1}$$

Lemma D.1 (approximation)**.** *Under Assumption D.1(b) there are $\delta_\iota^* \in \mathbb{R}^{\tilde{d}}$ with, for every component ι, $\|\beta_{ts,\iota} - \tilde{b}(\cdot)'\delta_\iota^*\|_{L^\infty(\mathcal{S})} \lesssim \tilde{K}_N^{-\chi}$ and $\|\partial_1\beta_{ts,\iota} - \partial_1\tilde{b}(\cdot)'\delta_\iota^*\|_{L^\infty(\mathcal{S})} \lesssim \tilde{K}_N^{-(\chi-1)}$.*

Proof. Apply the de Boor–Fix quasi-interpolant argument of Lemma B.4 to each $\beta_{ts,\iota} \in C^\chi$ with $\tilde{q} \ge \chi + 1$. The simultaneous bounds give the level rate $\tilde{K}_N^{-\chi}$ and the first-derivative rate $\tilde{K}_N^{-(\chi-1)}$ on the mesh covering $\mathcal{S}$. □

Lemma D.2 (design)**.** *Let $\mathbb{Q}_N := \mathbb{P}_N[T_i^t\varphi_i\varphi_i']$ and $\mathbb{Q} := \mathbb{E}[T^t\varphi\varphi']$, restricted to coefficients of basis functions meeting $\mathcal{S}^-$. Under Assumption D.1(a),(c),(d): (i) $c\,\tilde{h}^2\,\mathbf{I} \le \mathbb{Q} \le C\,\tilde{h}^2\,\mathbf{I}$; (ii) $\|\mathbb{Q}^{-1/2}(\mathbb{Q}_N - \mathbb{Q})\mathbb{Q}^{-1/2}\|_{\mathrm{op}} = O_{\mathrm{P}}(\tilde{K}_N\sqrt{\log N/N}) = o_{\mathrm{P}}(1)$; (iii) with generated regressors, $\|\mathbb{Q}^{-1/2}(\hat{\mathbb{Q}}_N - \mathbb{Q}_N)\mathbb{Q}^{-1/2}\|_{\mathrm{op}} = O_{\mathrm{P}}(\tilde{K}_N\Delta_N) = o_{\mathrm{P}}(1)$, where $\hat{\mathbb{Q}}_N := \mathbb{P}_N[T_i^t\hat{\varphi}_i\hat{\varphi}_i']$.*

Proof. (i) For a coefficient vector $c = (c_\iota)_\iota$,

$$c'\mathbb{Q}c = \mathbb{E}\left[T^t(\sum_\iota X_\iota\,\tilde{b}'c_\iota)^2\right] = \mathbb{E}[T^t\,g_c(S)'\mathbb{E}[XX'\,|\,T^t = 1, S]g_c(S)]$$

with $g_{c,\iota} := \tilde{b}'c_\iota$. Assumption D.1(a) floors the middle matrix on $\mathcal{S}$ and boundedness of X caps it, while the two-sided density of S on $\mathcal{S}$ and de Boor L^2-stability give $\|g_{c,\iota}\|^2_{L^2(dF_S)} \asymp \tilde{h}^2\|c_\iota\|^2$.

(ii) Matrix Bernstein (Tropp, 2015, Theorem 6.1.1): the normalized summands $\mathbb{Q}^{-1/2}T^t\varphi\varphi'\mathbb{Q}^{-1/2}/N$ have operator norm $\lesssim \tilde{d}^{\text{lev}}/N$ with leverage $\tilde{d}^{\text{lev}} \lesssim \tilde{K}_N^2$ (tensor splines in two dimensions with a density floor), giving the tail $\exp(-cNt^2/\tilde{K}_N^2)$ and the stated rate after the union bound over the $O(\tilde{K}_N^2)$-dimensional index. The condition $\tilde{K}_N^2 \log N = o(N)$ follows from the coupling in Assumption D.1(d), since $\rho_N \geq \sqrt{\log N/N}$.

(iii) Each entry of $\tilde{b}(\hat{S}_i) - \tilde{b}(S_i)$ is bounded by $\min(1, C\tilde{K}_N\Delta_N)$ and only $O(1)$ entries are active per observation, so $\|\hat{\varphi}_i - \varphi_i\| \lesssim \tilde{K}_N\Delta_N$ and $\|\varphi_i\| \lesssim 1$. An entry of $\hat{\mathbb{Q}}_N - \mathbb{Q}_N$ averages only over observations in the $O(1)$ cells met by both basis functions, and those cells carry empirical mass $O_{\mathrm{P}}(\tilde{h}^2)$ by the two-sided density of Assumption D.1(c), so the entry is $O_{\mathrm{P}}(\tilde{h}^2 \cdot \tilde{K}_N\Delta_N) = O_{\mathrm{P}}(\tilde{h}\Delta_N)$ and the band structure carries this to the operator norm. The two factors of $\mathbb{Q}^{-1/2} \asymp \tilde{h}^{-1}$ then give $O_{\mathrm{P}}(\tilde{K}_N\Delta_N)$, the quadratic term being smaller by a further $\tilde{K}_N\Delta_N$, and $\tilde{K}_N\Delta_N = O_{\mathrm{P}}(\tilde{K}_N\nu_N) = o_{\mathrm{P}}(1)$ by the coupling $\tilde{K}_N\nu_N \to 0$ in Assumption D.1(d). □

Lemma D.3 (oracle rates)**.** *Let $\tilde{\delta}$ solve the least squares problem with the true regressors S_i. Then, for every ι,*

$$\|\tilde{b}(\cdot)'\tilde{\delta}_\iota - \beta_{ts,\iota}\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}\Big(\tilde{K}_N\sqrt{\log N/N} + \tilde{K}_N^{-\chi}\Big),$$
$$\|\partial_1\tilde{b}(\cdot)'\tilde{\delta}_\iota - \partial_1\beta_{ts,\iota}\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}\Big(\tilde{K}_N^2\sqrt{\log N/N} + \tilde{K}_N^{-(\chi-1)}\Big).$$

Proof. Write $Y_i = \varphi_i'\delta^* + a_i + \varepsilon_i$ on $T_i^t = 1$, with $a_i := X_i'(\beta_{ts}(S_i) - (\tilde{b}'\delta_\iota^*)_\iota(S_i))$ the approximation error, $|a_i| \lesssim \tilde{K}_N^{-\chi}$ on $\mathcal{S}$. Then $\tilde{\delta} - \delta^* = \mathbb{Q}_N^+\mathbb{P}_N[T^t\varphi(\varepsilon + a)]$. For the variance part, fix $\zeta \in \mathcal{S}^-$: the fitted value $\tilde{b}(\zeta)'(\tilde{\delta}_\iota - \delta_\iota^*)$ is a linear contrast with Var $\lesssim \bar{\sigma}^2\,\tilde{b}(s)'[\mathbb{Q}^{-1}]_{pp}\tilde{b}(s)/N \lesssim \tilde{K}_N^2/N$ by Lemma D.2(i) and (ii). A Bernstein tail plus a union bound over an N^{-C}-net of $\mathcal{S}^-$ (the fitted spline is $\tilde{K}_N$-Lipschitz up to scale) gives the $\sqrt{\log N}$ factor and the sup-norm variance rate $\tilde{K}_N\sqrt{\log N/N}$. The bias part contributes $\|\tilde{b}'\mathbb{Q}_N^+\mathbb{P}_N[T^t\varphi a]\|_\infty \lesssim \tilde{K}_N^{-\chi}$ by the same contrast bound and $|a_i| \lesssim \tilde{K}_N^{-\chi}$. The derivative rates follow by the Markov inequality for splines, $\|\partial_1 g\|_{L^\infty} \leq C\tilde{K}_N\|g\|_{L^\infty}$ applied to the spline error, together with the derivative approximation bound of Lemma D.1.

□

Lemma D.4 (generated-regressor perturbation)**.**

$$\max_\iota \|\tilde{b}(\cdot)'(\hat{\delta}_\iota^{\text{ls}} - \tilde{\delta}_\iota)\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}(\nu_N) + o_{\mathrm{P}}\Big(\tilde{K}_N\sqrt{\log N/N}\Big).$$

Proof. The generated problem regresses Y_i on $\hat{\varphi}_i$. Decompose $Y_i = \hat{\varphi}_i'\delta^* + \hat{a}_i + b_i + \varepsilon_i$ with $\hat{a}_i$ the approximation error at $\hat{S}_i$ and $b_i := X_i'(\beta_{ts}(S_i) - \beta_{ts}(\hat{S}_i))$, $|b_i| \lesssim \Delta_N$ by the Lipschitz continuity of β_{ts} on $\mathcal{S}$. Then $\hat{\delta}^{\mathrm{ls}} - \tilde{\delta} = \hat{\mathbb{Q}}_N^+ \mathbb{P}_N[T^t \hat{\varphi}(\varepsilon + \hat{a} + b)] - \mathbb{Q}_N^+ \mathbb{P}_N[T^t \varphi(\varepsilon + a)]$, and the difference splits into (i) the mean-model error b: a contrast bounded by $C\Delta_N = O_{\mathrm{P}}(\nu_N)$; (ii) the design perturbation: by Lemma D.2(iii) a relative $o_{\mathrm{P}}(1)$ change, which multiplies the oracle variance rate; (iii) the noise cross term of the form $(\hat{\mathbb{Q}}_N^+ \mathbb{P}_N[T^t(\hat{\varphi} - \varphi)\varepsilon])$: the class $\{\tilde{b}_j(\cdot + \eta) : \|\eta\| \le C\nu_N\}$ is a fixed-dimensional Lipschitz-parametrized spline class, so the centered process is $O_{\mathrm{P}}(\tilde{K}_N\sqrt{\log N/N})$ times the modulus $\min(1, \tilde{K}_N\nu_N) = o_{\mathrm{P}}(1)$; (iv) the approximation-error shift $\hat{a} - a$, bounded by $\tilde{K}_N^{-\chi}$-level terms already counted. Summing gives the claim. □

Lemma D.5 (the first-stage variance scale)**.** *Suppose Assumptions 1, 3, 4 and 5 hold and the threshold of* $\hat{\Sigma}_\zeta := \widehat{\mathrm{Avar}}(\hat{\theta})$ *satisfies* $K_N\rho_N = o(\mathsf{tol}_N)$*, as in Assumption D.1(d). Then*

$$\omega_N := \frac{1}{N} \max_{i \le N:\, Z_i \in \mathcal{Z}_N} \left\| \hat{\dot{S}}_i \hat{\Sigma}_\zeta \hat{\dot{S}}_i' \right\|_{\mathrm{op}} = O_{\mathrm{P}}\left(\frac{K_N^2}{N}\right) = o_{\mathrm{P}}\left((K_N\rho_N)^2\right).$$

Proof. Write a sieve direction as $u = (\mathrm{vec}(V), a)$ with $V \in \mathcal{T}_{\boldsymbol{\gamma}_o}$, and let $\hat{I}_N$, I_N and $\hat{R}_N$ be as in Section C.2. By Section 4.2, $\hat{\Sigma}_\zeta$ is the inverse on $\hat{R}_N$ of $\hat{I}_N$ with its index block floored, and zero on the orthogonal complement of $\hat{R}_N$; the floor only raises eigenvalues, so $\hat{\Sigma}_\zeta$ is dominated by the inverse of $\hat{I}_N$ on $\hat{R}_N$, and for every vector ℓ of the same dimension

$$\ell'\hat{\Sigma}_\zeta\ell \le \sup\left\{\frac{(\ell'u)^2}{u'\hat{I}_N u} : u \in \hat{R}_N \setminus \{0\}\right\}.$$

Fix i with $Z_i \in \mathcal{Z}_N$, a unit vector $e \in \mathbb{R}^2$, and take $\ell = \hat{\dot{S}}_i'e$. By the form of $\hat{\dot{S}}_i$ in Section 4.2, $e_1'\hat{\dot{S}}_i u = \nabla P_{\hat{\xi}}^t(\hat{Z}_i)'V'W_i + \partial_{z^t}(b_{K_N}'a)(\hat{Z}_i)$ and $e_2'\hat{\dot{S}}_i u = (V^t - V^r)'W_i$. Since $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| = O_{\mathrm{P}}(\rho_N)$ and $\rho_N = o(K_N^{-1})$, the point $\hat{Z}_i$ lies, with probability tending to one and uniformly over such i, in a cell contained in $\mathcal{Z}$, on which the inverse inequality for polynomials gives $|\partial_{z^t}(b_{K_N}'a)(\hat{Z}_i)|^2 \le CK_N^2 \int_{\mathrm{cell}} \|\nabla(b_{K_N}'a)\|^2 dz \le CK_N^2 \|\nabla(b_{K_N}'a)\|_{L^2(\mathcal{Z})}^2$. With $\|DP_{\hat{\xi}}\| \le M$ on $\mathcal{P}$ and W_i bounded,

$$(\ell'u)^2 \le C\left(|||V|||^2 + K_N^2\|\nabla(b_{K_N}'a)\|_{L^2(\mathcal{Z})}^2\right).$$

For the denominator, $u'I_N u = \|\phi_V + \nabla(b_{K_N}'a)\|_{P_o}^2 \ge (|||V|||^2 + \|\nabla(b_{K_N}'a)\|_{L^2(\mathcal{Z})}^2)/C$ by Lemma C.18(ii), and $u'\hat{I}_N u = (1 + o_{\mathrm{P}}(1))u'I_N u$ uniformly over $u \in \hat{R}_N$ by Lemma C.18(iv).

Combining the two displays, $e'\hat{\dot{S}}_i\hat{\Sigma}_\zeta\hat{\dot{S}}_i'e \le CK_N^2(1+o_{\mathrm{P}}(1))$ uniformly over i with $Z_i \in \mathcal{Z}_N$ and over unit e, which is the bound on ω_N. Finally $K_N^2/N = o(K_N^2\rho_N^2)$ since $\rho_N^2 \ge K_N^2 \log N/N$. □

Lemma D.6 (errors-in-variables correction)**.** *Under the conditions of Lemma D.5 and Assumption D.1, with ω_N as there and $K_N\rho_N = O(\nu_N)$, for every ι,*

$$\|\tilde{b}(\cdot)'(\hat{\delta}_\iota - \hat{\delta}_\iota^{\mathrm{ls}})\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}(\tilde{K}_N^2\omega_N) = o_{\mathrm{P}}(\nu_N),$$
$$\|\partial_1\tilde{b}(\cdot)'(\hat{\delta}_\iota - \hat{\delta}_\iota^{\mathrm{ls}})\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}(\tilde{K}_N^3\omega_N) = o_{\mathrm{P}}(\tilde{K}_N\nu_N).$$

Proof. Write $\hat{\mathcal{M}}_1 := \min\{\hat{\mathcal{M}}, \mathbf{I}\}$, so that $\hat{\delta} - \hat{\delta}^{\mathrm{ls}} = \hat{\mathbb{Q}}_N^{-1/2}\hat{\mathcal{M}}_1\hat{\mathbb{Q}}_N^{1/2}\hat{\delta}^{\mathrm{ls}}$ by (19). Since $\hat{\Sigma}_\zeta$ is positive semidefinite, so are $\hat{\Sigma}_v$ and $\hat{\mathcal{M}}$, and $0 \preceq \hat{\mathcal{M}}_1 \preceq \hat{\mathcal{M}}$ because $\min\{\lambda, 1\} \le \lambda$ for $\lambda \ge 0$. Fix $\zeta \in \mathcal{S}^-$ and ι, and let $e_\iota \otimes \tilde{b}(\zeta)$ be the vector whose inner product with a coefficient vector c returns $\tilde{b}(\zeta)'c_\iota$. With $u := \hat{\mathbb{Q}}_N^{-1/2}(e_\iota \otimes \tilde{b}(\zeta))$, $v := \hat{\mathbb{Q}}_N^{1/2}\hat{\delta}^{\mathrm{ls}}$ and $q := \hat{\mathbb{Q}}_N^{-1}(e_\iota \otimes \tilde{b}(\zeta))$,

$$|\tilde{b}(\zeta)'(\hat{\delta}_\iota - \hat{\delta}_\iota^{\mathrm{ls}})| = |u'\hat{\mathcal{M}}_1 v| \le (u'\hat{\mathcal{M}}u)^{1/2}(v'\hat{\mathcal{M}}v)^{1/2} = (q'\hat{\Sigma}_v q)^{1/2}(\hat{\delta}^{\mathrm{ls}\prime}\hat{\Sigma}_v\hat{\delta}^{\mathrm{ls}})^{1/2},$$

by the Cauchy–Schwarz inequality for the positive semidefinite $\hat{\mathcal{M}}_1$ and $\hat{\mathcal{M}}_1 \preceq \hat{\mathcal{M}}$.

The two quadratic forms are bounded through the form of $\hat{\Sigma}_v$ in Section 4.2. For a coefficient vector $c = (c_{\iota'})_{\iota'}$ let $g_c(s,x) := \sum_{\iota'} x_{\iota'}\tilde{b}(s)'c_{\iota'}$, so that $(X_i \otimes D\tilde{b}(\hat{S}_i)\hat{\dot{S}}_i)'c = \hat{\dot{S}}_i'\nabla_s g_c(\hat{S}_i, X_i)$ and

$$\begin{aligned} c'\hat{\Sigma}_v c &= \frac{1}{N}\mathbb{P}_N\left[T^t\,\nabla_s g_c(\hat{S},X)'\left(\hat{\dot{S}}\hat{\Sigma}_\zeta\hat{\dot{S}}'\right)\nabla_s g_c(\hat{S},X)\right] \\ &\le \omega_N\,\mathbb{P}_N\left[T^t\|\nabla_s g_c(\hat{S},X)\|^2\right], \end{aligned}$$

with ω_N as in Lemma D.5, since $\nabla_s g_c(\hat{S}_i, X_i)$ vanishes unless $S_i \in \mathcal{S}$, and $S_i \in \mathcal{S}$ implies $Z_i \in \mathcal{Z}_N$ for N large. The right side is a quadratic form in c whose Gram matrix is that of the derivative basis $X \otimes D\tilde{b}(\hat{S})$ on the t-subsample. The argument of Lemma D.2(i)–(iii) applied to this basis shows that it is comparable, up to a factor $1+o_{\mathrm{P}}(1)$, to its population counterpart under Lebesgue measure on the cells, where the Markov inequality for splines bounds the gradient Gram by $C\tilde{K}_N^2$ times the level Gram. Hence

$$\mathbb{P}_N\left[T^t\|\nabla_s g_c(\hat{S},X)\|^2\right] \le C\tilde{K}_N^2\,c'\hat{\mathbb{Q}}_N c\,(1+o_{\mathrm{P}}(1)) \tag{D.2}$$

uniformly in c.

For $c = q$, (D.2) and $q'\hat{\mathbb{Q}}_N q = \tilde{b}(\zeta)'[\hat{\mathbb{Q}}_N^{-1}]_{pp}\tilde{b}(\zeta) \lesssim \tilde{K}_N^2$, by Lemma D.2, give $q'\hat{\Sigma}_v q = O_{\mathrm{P}}(\omega_N \tilde{K}_N^4)$ uniformly in $\zeta \in \mathcal{S}^-$. For $c = \hat{\delta}^{\mathrm{ls}}$, write $\hat{\delta}^{\mathrm{ls}} = \delta^* + (\hat{\delta}^{\mathrm{ls}} - \delta^*)$ with δ^* the approximant of Lemma D.1. The quasi-interpolant of Lemma B.4 approximates the gradient of β_{ts} as well as its level, so $\|\nabla_s g_{\delta^*}(\cdot, x)\|_{L^\infty(\mathcal{S})} \lesssim 1$ uniformly in bounded x. For the estimation error, the arguments of Lemmas D.3 and D.4 in the norm of $\hat{\mathbb{Q}}_N$ give

$$\|\hat{\delta}^{\mathrm{ls}} - \delta^*\|_{\hat{\mathbb{Q}}_N} = O_{\mathrm{P}}\left(\tilde{K}_N/\sqrt{N} + \tilde{K}_N^{-\chi} + \nu_N\right),$$

the variance part because $\mathbb{E}\|\mathbb{Q}^{-1/2}\mathbb{P}_N[T^t\varphi\varepsilon]\|^2 \le \bar{\sigma}^2\tilde{d}\,\dim(X)/N \lesssim \tilde{K}_N^2/N$, and the approximation part and the generated-regressor part because a function bounded by $\tilde{K}_N^{-\chi}$ or by $C\Delta_N$ has a least-squares projection of no larger norm. By (D.2) applied to $c = \hat{\delta}^{\mathrm{ls}} - \delta^*$,

$$\mathbb{P}_N\left[T^t\|\nabla_s g_{\hat{\delta}^{\mathrm{ls}}}(\hat{S}, X)\|^2\right] \lesssim 1 + \tilde{K}_N^2\left(\tilde{K}_N^2/N + \tilde{K}_N^{-2\chi} + \nu_N^2\right)(1 + o_{\mathrm{P}}(1)) = O_{\mathrm{P}}(1),$$

since $\tilde{K}_N^2 \log N = o(N)$ as noted in the proof of Lemma D.2(ii), $\chi \ge 2$, and $\tilde{K}_N\nu_N \to 0$ by Assumption D.1(d). Hence $\hat{\delta}^{\mathrm{ls}\prime}\hat{\Sigma}_v\hat{\delta}^{\mathrm{ls}} = O_{\mathrm{P}}(\omega_N)$.

Combining the two bounds, $\sup_{\zeta\in\mathcal{S}^-}|\tilde{b}(\zeta)'(\hat{\delta}_\iota - \hat{\delta}_\iota^{\mathrm{ls}})| = O_{\mathrm{P}}((\omega_N\tilde{K}_N^4)^{1/2}\omega_N^{1/2}) = O_{\mathrm{P}}(\tilde{K}_N^2\omega_N)$. The derivative bound follows from the Markov inequality for splines, $\|\partial_1 g\|_{L^\infty} \le C\tilde{K}_N\|g\|_{L^\infty}$, applied to $g = \tilde{b}(\cdot)'(\hat{\delta}_\iota - \hat{\delta}_\iota^{\mathrm{ls}})$. Finally, $\omega_N = o_{\mathrm{P}}((K_N\rho_N)^2) = o_{\mathrm{P}}(\nu_N^2)$ by Lemma D.5 and $K_N\rho_N = O(\nu_N)$, and $\tilde{K}_N^2\nu_N^2 = (\tilde{K}_N\nu_N)\nu_N = o_{\mathrm{P}}(\nu_N)$ and $\tilde{K}_N^3\nu_N^2 = (\tilde{K}_N\nu_N)\tilde{K}_N\nu_N = o_{\mathrm{P}}(\tilde{K}_N\nu_N)$ by the coupling $\tilde{K}_N\nu_N \to 0$. □

Proof of Theorem D.1. Combining Lemmas D.3 and D.4,

$$\|\tilde{b}(\cdot)'\hat{\delta}_\iota^{\mathrm{ls}} - \beta_{ts,\iota}\|_{L^\infty(\mathcal{S}^-)} = O_{\mathrm{P}}\Big(\tilde{K}_N\sqrt{\log N/N} + \tilde{K}_N^{-\chi} + \nu_N\Big),$$

and applying the spline Markov inequality to the estimation error plus Lemma D.1 to the bias, the derivative rate carries one extra factor $\tilde{K}_N$ in all three terms, with the passthrough term $\tilde{K}_N\nu_N$. Theorem 2(ii) supplies Assumption D.1(e) with $\nu_N = K_N\rho_N$, and with this ν_N Lemma D.6 applies. By that lemma, $\hat{\beta}_{ts,\iota} = \tilde{b}(\cdot)'\hat{\delta}_\iota$ differs from $\tilde{b}(\cdot)'\hat{\delta}_\iota^{\mathrm{ls}}$ by $o_{\mathrm{P}}(\nu_N)$ in the level and by $o_{\mathrm{P}}(\tilde{K}_N\nu_N)$ in the propensity derivative, uniformly on $\mathcal{S}^-$, so the same rates hold for $\hat{\beta}_{ts,\iota}$ and $\widehat{\partial_1\beta}_{ts,\iota}$. For the functionals: at a point (z, x) whose arguments lie in $\mathcal{S}^-$, $\hat{\mu}_{ts}(z, x) - \mu_{ts}(z, x)$ is a fixed linear combination, with bounded weights, of $\hat{\beta}_{ts}(\hat{P}^t(z), z^{t,r}) - \beta_{ts}(P_o^t(z), z^{t,r})$ and the corresponding derivative difference weighted by $\hat{P}^t(z)$. Each difference splits into the estimation error of the fitted function at the estimated

argument, bounded by the uniform rates above, and the movement of the argument, bounded by the Lipschitz continuity of β_{ts} and $\partial_1\beta_{ts}$ times $|\hat{P}^t(z) - P_o^t(z)| = O_{\mathrm{P}}(\nu_N)$. The derivative difference dominates. The same holds for $\hat{\tau}_{ts}$ with the roles of (t, s) reversed and the derivative terms included, giving the stated uniform rates at the derivative order. The Lipschitz continuity of $\partial_1\beta_{ts}$ uses $\chi \geq 2$, and the bounded weights use $|\hat{P}^t| \leq 1$ together with boundedness of x. The evaluation of $\hat{P}^t(z)$ at nonrandom z invokes Assumption D.1(e) on the region carrying the suprema. □

E Inference on the Aggregate Effects

This section proves Theorem 4. Fix the margin (t, s) and the remaining treatment r, and use the notation of Section D for the regression of β_{ts} on the t-subsample, namely $S_i = S_i^{t,r}$, $\hat{S}_i$, $\tilde{b}$, $\tilde{h}$, φ_i, $\hat{\varphi}_i$, $\mathcal{S}^-$, $\mathcal{S}$ and Δ_N of (D.1), with $\nu_N = K_N\rho_N$. Assumption 6(b) places the compact set $\mathsf{S}^t(\mathcal{Z}')$ in the interior of $\mathcal{S}$, and we take $\mathcal{S}^-$ to contain it in its interior. Here $\hat{\mathbb{Q}}_N := \mathbb{P}_N[T_i^t\hat{\varphi}_i\hat{\varphi}_i']$ is the Gram matrix of all basis functions, as in Section 4.2, and $\hat{\mathbb{Q}}_N^+$ is its Moore–Penrose inverse. Assumption 6 contains Assumption D.1(a), (b) and the mesh condition of (c), its part (d) implies Assumption D.1(d), and Theorem 2(ii) gives Assumption D.1(e). With the density bounds derived next, the lemmas of Section D cited below therefore apply.

Under Assumption 1(c), $\det D\mathsf{S}^t(z)$ is, up to a nonzero factor, the derivative of P_o^t at z in the direction that keeps $z^{t,r}$ fixed. That derivative is an integral of f over a half-line, and it has no zeros since f is positive. With the injectivity of S^t (Section 5.2), S^t is a C^1-diffeomorphism[12] of $\mathbb{R}^2$ onto an open set, and the density $p_Z(z)/|\det D\mathsf{S}^t(z)|$ of S_i at $\mathsf{S}^t(z)$ is bounded above and below on compact subsets of the interior of $\mathrm{supp}(S_i)$ by Assumption 4(d). The generated regressors obey

$$\max_{i\leq N}\|\hat{S}_i - S_i\| \leq C\left(\max_{i\leq N}\|\hat{Z}_i - Z_i\| + \|P_{\hat{\xi}} - P_o\|_{L^\infty(\mathcal{Z}_N)} + \sup_{z\in\mathcal{Z}}\mathrm{dist}(z, \mathcal{Z}_N)\right) \tag{E.1}$$

by the Lipschitz bound on $\mathcal{P}$, and the right side is $O_{\mathrm{P}}(K_N\rho_N + K_N^{-1})$ by Theorem 2(ii) and the boundary of $\mathcal{Z}$ of class $C^{2,\alpha}$ in Assumption 5(b).

For a coefficient vector $c = (c_\iota)_\iota$, write $g_c := (\tilde{b}'c_\iota)_\iota$, so that $\hat{\varphi}_i'c = X_i'g_c(\hat{S}_i)$. For a

[12] For two open subsets of $\mathbb{R}^2$, a C^k-diffeomorphism of one onto the other is a bijection between them such that the map and its inverse are both of class C^k.

coefficient function $g = (g_\iota)_\iota$ of class C^1 on a neighborhood of $\mathsf{S}^t(\mathcal{Z}_\upsilon)$, let

$$\mathsf{A}^t(g) := \int_{\mathbb{R}^2} \upsilon(z) x_0' \partial_1(pg)(\mathsf{S}^t(z)) dz, \qquad \hat{\mathsf{A}}^t(g) := \int_{\mathbb{R}^2} \upsilon(z) x_0' \partial_1(pg)(\hat{\mathsf{S}}^t(z)) dz, \tag{E.2}$$

where $\partial_1(pg)(\zeta) = g(\zeta) + p\partial_1 g(\zeta)$ with p the first coordinate of ζ. Then $\bar{\tau}_{ts,o} = \mathsf{A}^t(\beta_{ts}) - \mathsf{A}^s(\beta_{st})$ and $\bar{\tau}_{ts} = \hat{\mathsf{A}}^t(\hat{\beta}_{ts}) - \hat{\mathsf{A}}^s(\hat{\beta}_{st})$, with A^s and $\hat{\mathsf{A}}^s$ defined from S^s and $\hat{\mathsf{S}}^s$. Let L_N be the vector with $\mathsf{A}^t(g_c) = L_N' c$ for every c, and let

$$\hat{\mathcal{R}}_N(\zeta, x) := (x \otimes \tilde{b}(\zeta))' \hat{\mathbb{Q}}_N^+ L_N, \qquad \eta_N := \tilde{K}_N^{-(1+\alpha)} + \tilde{K}_N \sqrt{\log N / N} + \tilde{K}_N \nu_N. \tag{E.3}$$

The proof turns on one fact. Because the weight vanishes on the boundary of its support, integration by parts in the propensity coordinate writes A^t as the inner product with the fixed function $X'\mathcal{R}^t(S)$ in $L^2(T^t)$ (Lemma E.1). The least-squares fit then turns $\mathsf{A}^t(\hat{\beta}_{ts}^{\mathrm{ls}})$ into an average of the residuals weighted by $\hat{\mathcal{R}}_N$, the projection of $X'\mathcal{R}^t(S)$ on the sieve at the generated regressors. The key challenge is that the fit is global, and $\hat{\mathbb{Q}}_N$ has no lower bound on the cells that the boundary of $\mathrm{supp}(S_i)$ crosses. Lemma E.2 resolves it with the support of the weight inside $\mathcal{S}^-$: those cells enter $\hat{\mathcal{R}}_N$ only through the coefficients of the basis functions on the boundary layer of $\mathcal{S}^-$, where $\mathcal{R}^t$ vanishes. The remaining steps bound the outcome, approximation and generated-regressor terms with $X'\mathcal{R}^t(S)$ in place of $\hat{\mathcal{R}}_N$. The correction changes the aggregate by $O_{\mathrm{P}}(K_N^2/N)$, and the evaluation of the fitted functions at $\hat{\mathsf{S}}^t(z)$ becomes a regular functional of the first stage.

Lemma E.1 (representer of the aggregate)**.** *Under Assumptions 1, 4(b),(d), 5(b) and 6(a)–(c), the function $\mathcal{R}^t$ of (20) is of class $C^{1,\alpha}$ on $\mathbb{R}^2$ and vanishes outside $\mathsf{S}^t(\mathcal{Z}_\upsilon)$, and for every g of class C^1 on a neighborhood of $\mathsf{S}^t(\mathcal{Z}_\upsilon)$,*

$$\mathsf{A}^t(g) = \mathbb{E}\left[T_i^t X_i' \mathcal{R}^t(S_i) X_i' g(S_i)\right]. \tag{E.4}$$

Proof. By the beginning of this section, $|\det D\mathsf{S}^t|$ is bounded below on a neighborhood of the compact set $\mathcal{Z}_\upsilon$, and $\mathcal{Z}_\upsilon \subset (\mathsf{S}^t)^{-1}(\mathrm{Int}\,\mathcal{S}) \subset \mathrm{Int}\,\mathcal{Z}$ by Assumption 6(b). The change of variables $\zeta = \mathsf{S}^t(z)$ gives $\mathsf{A}^t(g) = \int \upsilon^t(\zeta) x_0' \partial_1(pg)(\zeta) d\zeta$. The function υ^t vanishes on the boundary of $\mathsf{S}^t(\mathcal{Z}_\upsilon)$, and integration by parts in p leaves no boundary term,

$$\mathsf{A}^t(g) = -\int_{\mathbb{R}^2} \partial_1 \upsilon^t(\zeta) p x_0' g(\zeta) d\zeta.$$

Given W_i, the treatment is t with probability $P_o^t(Z_i)$, and (Z_i, X_i) is a function of W_i.

Hence $\mathbb{E}[T_i^t X_i X_i' \,|\, Z_i = z] = P_o^t(z)\mathbb{E}[X_i X_i' \,|\, Z_i = z]$, and the change of variables gives, with $z = (\mathsf{S}^t)^{-1}(\zeta)$,

$$\mathbb{E}\left[T_i^t X_i' \mathcal{R}^t(S_i) X_i' g(S_i)\right] = \int_{\mathbb{R}^2} \frac{p_Z(z)}{|\det D\mathsf{S}^t(z)|} p\mathcal{R}^t(\zeta)' \mathbb{E}[X_i X_i' \,|\, Z_i = z] g(\zeta) d\zeta.$$

By (20), the integrand equals $-\partial_1 v^t(\zeta) p x_0' g(\zeta)$, which proves (E.4). For the smoothness, $k_o \geq 3 + \alpha$ makes S^t and its inverse of class $C^{3,\alpha}$ near $\mathcal{Z}_v$ and $\mathsf{S}^t(\mathcal{Z}_v)$, and $v \in C^{2,1}$ then makes v^t of class $C^{2,\alpha}$ with its first two derivatives vanishing on the boundary of $\mathsf{S}^t(\mathcal{Z}_v)$. The function $\partial_1 v^t$, extended by zero, is therefore of class $C^{1,\alpha}$. The density p_Z is of class $C^{1,\alpha}$ by Assumption 5(b) and bounded below by Assumption 4(d), and $\mathbb{E}[X_i X_i' \,|\, Z_i = z]$ is of class $C^{1,\alpha}$ near $\mathcal{Z}_v$ by Assumption 6(c). Its smallest eigenvalue is bounded below near $\mathcal{Z}_v$, since $\mathbb{E}[X_i X_i' \,|\, S_i, T_i = t] = \mathbb{E}[X_i X_i' \,|\, Z_i]$ by the same factorization and the design condition of Assumption 6(a) holds on $\{S_i \in \mathcal{S}\}$. □

Lemma E.2 (localization of the sample representer)**.** *Under Assumptions 1, 3, 4, 5 and 6,*

$$\mathbb{P}_N\left[T_i^t \left(\hat{\mathcal{R}}_N(\hat{S}_i, X_i) - X_i' \mathcal{R}^t(S_i)\right)^2\right] = O_{\mathrm{P}}(\eta_N^2). \tag{E.5}$$

Proof. The fact behind the lemma is that L_N loads only basis functions whose supports meet $\mathsf{S}^t(\mathcal{Z}_v)$, at a fixed distance inside $\mathcal{S}^-$, while the cells on which $\hat{\mathbb{Q}}_N$ has no lower bound lie outside $\mathcal{S}$.

Partition. Let $\mathcal{S}_N$ be the union of the mesh cells contained in $\mathcal{S}^-$, let $\mathcal{I}$ index the basis functions whose supports contain a cell of $\mathcal{S}_N$, and let $\mathcal{I}_\partial \subseteq \mathcal{I}$ index those whose supports also contain a cell outside $\mathcal{S}_N$. The support of a basis function has diameter at most $C\tilde{h}$, and a cell outside $\mathcal{S}_N$ contains a point outside $\mathcal{S}^-$. For $\tilde{K}_N$ large, the supports of the functions in $\mathcal{I}$ therefore lie in $\mathcal{S}$, and those of the functions in $\mathcal{I}_\partial$ lie outside a fixed neighborhood of $\mathsf{S}^t(\mathcal{Z}_v)$. By Lemma E.1, L_N vanishes outside $\mathcal{I} \setminus \mathcal{I}_\partial$. Split $\hat{\mathbb{Q}}_N = \hat{\mathbb{Q}}_N^{\mathrm{in}} + \hat{\mathbb{Q}}_N^{\mathrm{out}}$ with $\hat{\mathbb{Q}}_N^{\mathrm{in}} := \mathbb{P}_N[T_i^t \hat{\varphi}_i \hat{\varphi}_i' \mathbb{1}\{\hat{S}_i \in \mathcal{S}_N\}]$, and let $\mathbb{Q}^{\mathrm{in}} := \mathbb{E}[T^t \varphi \varphi' \mathbb{1}\{S \in \mathcal{S}_N\}]$. A case with $\hat{S}_i \in \mathcal{S}_N$ activates only basis functions in $\mathcal{I}$, and a case with $\hat{S}_i \notin \mathcal{S}_N$ activates none in $\mathcal{I} \setminus \mathcal{I}_\partial$. Hence $\hat{\mathbb{Q}}_N^{\mathrm{in}}$ and $\mathbb{Q}^{\mathrm{in}}$ vanish outside the block $\mathcal{I} \times \mathcal{I}$, and $\hat{\mathbb{Q}}_N^{\mathrm{out}}$ vanishes on every row and column in $\mathcal{I} \setminus \mathcal{I}_\partial$.

The inner representer. The cells of $\mathcal{S}_N$ lie in $\mathcal{S}$, where the density bounds and the design condition hold, and every function in $\mathcal{I}$ has a cell of $\mathcal{S}_N$ in its support. The proof of Lemma D.2, applied to the cases in $\mathcal{S}_N$ with the local L^2 stability of the B-spline basis on these cells, gives $C^{-1}\tilde{h}^2 \mathbf{I} \leq \mathbb{Q}^{\mathrm{in}} \leq C\tilde{h}^2 \mathbf{I}$ on $\mathcal{I}$ and $\|(\mathbb{Q}^{\mathrm{in}})^{-1/2}(\hat{\mathbb{Q}}_N^{\mathrm{in}} - \mathbb{Q}^{\mathrm{in}})(\mathbb{Q}^{\mathrm{in}})^{-1/2}\|_{\mathrm{op}} =$

$O_{\mathrm{P}}(\eta_N)$. The cases that cross the boundary of $\mathcal{S}_N$ between S_i and $\hat{S}_i$ have $S_i \in \mathcal{S}$ by (E.1), lie within Δ_N of that boundary, and contribute $O_{\mathrm{P}}(\tilde{K}_N \Delta_N)$ to the same norm. Let $c^{\mathrm{in}} := (\mathbb{Q}^{\mathrm{in}})^{-1} L_N$ and $\hat{c}^{\mathrm{in}} := (\hat{\mathbb{Q}}_N^{\mathrm{in}})^{-1} L_N$, with the inverses taken on $\mathcal{I}$ and both vectors extended by zero. Since $\mathcal{R}^t$ vanishes outside $\mathcal{S}_N$, Lemma E.1 makes $X' g_{c^{\mathrm{in}}}(S)$ the projection of $X'\mathcal{R}^t(S)$ on $\{X' g_c(S) : c \text{ vanishes outside } \mathcal{I}\}$ in $L^2(T^t \mathbb{1}\{S \in \mathcal{S}_N\})$. The de Boor–Fix quasi-interpolant of $\mathcal{R}^t$ lies in this span for $\tilde{K}_N$ large and approximates $\mathcal{R}^t \in C^{1,\alpha}$ uniformly to $\tilde{K}_N^{-(1+\alpha)}$, and therefore

$$\mathbb{E}\left[T^t \mathbb{1}\{S \in \mathcal{S}_N\} \left(X' g_{c^{\mathrm{in}}}(S) - X'\mathcal{R}^t(S)\right)^2\right] \lesssim \tilde{K}_N^{-2(1+\alpha)}. \tag{E.6}$$

Moreover $\hat{c}^{\mathrm{in}} - c^{\mathrm{in}} = -(\hat{\mathbb{Q}}_N^{\mathrm{in}})^{-1}(\hat{\mathbb{Q}}_N^{\mathrm{in}} - \mathbb{Q}^{\mathrm{in}}) c^{\mathrm{in}}$ and $c^{\mathrm{in}\prime} \mathbb{Q}^{\mathrm{in}} c^{\mathrm{in}} \le \mathbb{E}[T^t (X'\mathcal{R}^t(S))^2]$, which give $(\hat{c}^{\mathrm{in}} - c^{\mathrm{in}})' \mathbb{Q}^{\mathrm{in}} (\hat{c}^{\mathrm{in}} - c^{\mathrm{in}}) = O_{\mathrm{P}}(\eta_N^2)$.

The interface coefficients. For $m \in \mathcal{I}_\partial$, let $\mathcal{S}_{N,m}$ be a cell of $\mathcal{S}_N$ in the support of the mth basis function. It lies within $C\tilde{h}$ of the complement of $\mathcal{S}^-$, where $\mathcal{R}^t = 0$ for $\tilde{K}_N$ large. The de Boor dual functionals bound a coefficient of a spline by $C\tilde{h}^{-1}$ times its L^2 norm on any one cell of the support of the basis function, and on $\mathcal{S}_{N,m}$ the density floor and the design condition bound that norm by $C\mathbb{E}[T^t (X' g_c(S))^2 \mathbb{1}\{S \in \mathcal{S}_{N,m}\}]^{1/2}$. Each cell is $\mathcal{S}_{N,m}$ for a bounded number of m, and (E.6) gives

$$\tilde{h}^2 \sum_{m \in \mathcal{I}_\partial} |c_m^{\mathrm{in}}|^2 \lesssim \mathbb{E}\left[T^t \mathbb{1}\{S \in \mathcal{S}_N\} \left(X' g_{c^{\mathrm{in}}}(S) - X'\mathcal{R}^t(S)\right)^2\right] \lesssim \tilde{K}_N^{-2(1+\alpha)}, \tag{E.7}$$

together with $\tilde{h}^2 \sum_{m \in \mathcal{I}_\partial} |\hat{c}_m^{\mathrm{in}} - c_m^{\mathrm{in}}|^2 = O_{\mathrm{P}}(\eta_N^2)$ from the previous paragraph.

Comparison with the global fit. With probability tending to one, $\hat{\mathbb{Q}}_N^{\mathrm{in}}$ is nonsingular on $\mathcal{I}$, every null vector of $\hat{\mathbb{Q}}_N$ vanishes on $\mathcal{I}$, and L_N lies in the range of $\hat{\mathbb{Q}}_N$. Since $\hat{\mathbb{Q}}_N \hat{c}^{\mathrm{in}} = L_N + \hat{\mathbb{Q}}_N^{\mathrm{out}} \hat{c}^{\mathrm{in}}$, the difference $\hat{c}^{\mathrm{out}} := \hat{\mathbb{Q}}_N^{+} L_N - \hat{c}^{\mathrm{in}}$ satisfies $\hat{\mathbb{Q}}_N \hat{c}^{\mathrm{out}} = -\hat{\mathbb{Q}}_N^{\mathrm{out}} \hat{c}^{\mathrm{in}}$, and $0 \preceq \hat{\mathbb{Q}}_N^{\mathrm{out}} \preceq \hat{\mathbb{Q}}_N$ gives

$$\mathbb{P}_N\left[T_i^t (\hat{\varphi}_i' \hat{c}^{\mathrm{out}})^2\right] = \hat{c}^{\mathrm{in}\prime} \hat{\mathbb{Q}}_N^{\mathrm{out}} \hat{\mathbb{Q}}_N^{+} \hat{\mathbb{Q}}_N^{\mathrm{out}} \hat{c}^{\mathrm{in}} \le \hat{c}^{\mathrm{in}\prime} \hat{\mathbb{Q}}_N^{\mathrm{out}} \hat{c}^{\mathrm{in}}.$$

The right side involves only the coefficients of $\hat{c}^{\mathrm{in}}$ in $\mathcal{I}_\partial$. The cells met by the functions in $\mathcal{I}_\partial$ lie in $\mathcal{S}$, and each carries empirical mass $O_{\mathrm{P}}(\tilde{h}^2)$ uniformly, by (E.1), $\Delta_N = O_{\mathrm{P}}(\nu_N)$ and $\tilde{K}_N^2 \log N = o(N)$. Each case activates a bounded number of basis functions, each bounded by one, and the right side is therefore $O_{\mathrm{P}}(\tilde{h}^2 \sum_{m \in \mathcal{I}_\partial} |\hat{c}_m^{\mathrm{in}}|^2) = O_{\mathrm{P}}(\eta_N^2)$ by (E.7).

Conclusion. Write $\hat{\mathcal{R}}_N(\hat{S}_i, X_i) = \hat{\varphi}_i' \hat{c}^{\mathrm{in}} + \hat{\varphi}_i' \hat{c}^{\mathrm{out}}$. On the cases with $\hat{S}_i \in \mathcal{S}_N$, split $\hat{\varphi}_i' \hat{c}^{\mathrm{in}} -$

$X_i'\mathcal{R}^t(S_i)$ into $\hat{\varphi}_i'(\hat{c}^{\text{in}}-c^{\text{in}})$, $(\hat{\varphi}_i-\varphi_i)'c^{\text{in}}$ and $\varphi_i'c^{\text{in}}-X_i'\mathcal{R}^t(S_i)$. The empirical second moments of the first two are $O_{\text{P}}(\eta_N^2)$ and, by the cell-local Markov inequality of Lemma D.2(iii), $O_{\text{P}}(\tilde{K}_N^2\Delta_N^2)$. For the third, the cases with $S_i \in \mathcal{S}_N$ contribute $O_{\text{P}}(\tilde{K}_N^{-2(1+\alpha)})$ by (E.6) and Markov's inequality. The cases with $S_i \notin \mathcal{S}_N$ have $S_i \in \mathcal{S}$ by (E.1) and lie within Δ_N of the boundary of $\mathcal{S}_N$, where $\mathcal{R}^t(S_i) = 0$. On them $\varphi_i'c^{\text{in}}$ involves only the coefficients in $\mathcal{I}_\partial$, and their empirical mass $O_{\text{P}}(\tilde{h}\Delta_N + \tilde{h}^2)$ on the support of each basis function bounds their contribution by $O_{\text{P}}(\tilde{h}^2\sum_{m\in\mathcal{I}_\partial}|c_m^{\text{in}}|^2)$. On the cases with $\hat{S}_i \notin \mathcal{S}_N$, the value $\hat{\varphi}_i'\hat{c}^{\text{in}}$ involves only the coefficients in $\mathcal{I}_\partial$ and is bounded as in the previous paragraph. These cases have $\mathcal{R}^t(S_i) = 0$ for N large, by (E.1) and the fixed distance between $\mathsf{S}^t(\mathcal{Z}_\upsilon)$ and the complement of $\mathcal{S}_N$. With $\Delta_N = O_{\text{P}}(\nu_N)$, (E.7) and the bound on $\hat{\varphi}_i'\hat{c}^{\text{out}}$, these bounds give (E.5). □

The first-stage functional of Theorem 4 has two parts on each side of the margin, one for the evaluation of the coefficient functions at the estimated propensity and one for the generated regressors in the fit. For $\theta = (\boldsymbol{\gamma}, P)$, let

$$\begin{aligned}
\Lambda^{\text{eval},t}(\theta) &:= \int_{\mathbb{R}^2} \upsilon(z)x_0'\partial_1(p\beta_{ts})(P^t(z), z^{t,r})dz, \\
\Lambda^{\text{reg},t}(\theta) &:= \mathbb{E}\left[T^tX'\mathcal{R}^t(S^{t,r})X'\beta_{ts}(S^{t,r}(\theta))\right], \\
\Lambda_{ts}^{\text{eval}} &:= \Lambda^{\text{eval},t} - \Lambda^{\text{eval},s}, \qquad \Lambda_{ts}^{\text{reg}} := \Lambda^{\text{reg},t} - \Lambda^{\text{reg},s}, \qquad \Lambda_{ts} := \Lambda_{ts}^{\text{eval}} - \Lambda_{ts}^{\text{reg}},
\end{aligned} \tag{E.8}$$

with $S^{t,r}(\theta)$ as in (15), the expectation over (T, W) at the true distribution, and $\Lambda^{\text{eval},s}$ and $\Lambda^{\text{reg},s}$ defined from β_{st}, $\mathcal{R}^s$ and $S^{s,r}$. The propensity loading of $\Lambda^{\text{eval},t}$ on P^t is $\upsilon(z)x_0'\partial_{11}(p\beta_{ts})(\mathsf{S}^t(z))$, and its index loading is zero. The derivative of $\Lambda^{\text{reg},t}$ in the direction (V, u) is $\mathbb{E}[T^tX'\mathcal{R}^t(S)X'\nabla\beta_{ts}(S)\dot{S}[V,u]]$, where $\dot{S}[V,u] := (\nabla P_o^t(Z)'V'W + u^t(Z), (e_t - e_r)'V'W)$ and e_t selects z^t, with $e_0 := 0$. Its propensity loading on P^t is $-P_o^t(z)|\det D\mathsf{S}^t(z)|\partial_1\upsilon^t(\mathsf{S}^t(z))x_0'\partial_1\beta_{ts}(\mathsf{S}^t(z))$, in which p_Z and $\mathbb{E}[X_iX_i'\,|\,Z_i = z]$ cancel. A loading on P^0 acts on (P^1, P^2) through $P^0 = 1 - P^1 - P^2$. Under Assumption 6(c), the loadings are of class $C^{1,\alpha}$, and the remainders of Definition 3(ii) are $O(\rho_N^2 + K_N\rho_N|||\boldsymbol{\gamma}-\boldsymbol{\gamma}_o|||)$ on the sieve ball, which is $o(N^{-1/2} + |||\boldsymbol{\gamma} - \boldsymbol{\gamma}_o|||)$ since $\sqrt{N}\rho_N^2 \to 0$ under Assumption 5(a). The four functionals, and hence Λ_{ts}, are regular.

The variance estimator is

$$\begin{aligned}
\hat{\sigma}_{ts}^2 := \mathbb{P}_N\left[T_i^t\mathbb{1}\{\hat{Z}_i \in \mathcal{Z}'\}\check{\mathcal{R}}_N^t(\hat{S}_i^{t,r}, X_i)^2\hat{\varepsilon}_{t,i}^2\right] \\
+ \mathbb{P}_N\left[T_i^s\mathbb{1}\{\hat{Z}_i \in \mathcal{Z}'\}\check{\mathcal{R}}_N^s(\hat{S}_i^{s,r}, X_i)^2\hat{\varepsilon}_{s,i}^2\right] + \hat{\sigma}_{\Lambda_{ts}}^2,
\end{aligned} \tag{E.9}$$

with $\mathcal{Z}'$ the trimming rectangle of Section 4.2. Here $\check{\mathcal{R}}_N^t$ is $\hat{\mathcal{R}}_N$ of (E.3) with L_N replaced by the vector $\hat{L}_N$ of $\hat{\mathsf{A}}^t$, $\hat{\varepsilon}_{t,i} := Y_i - X_i'\hat{\beta}_{ts}(\hat{S}_i^{t,r})$ is the residual of the corrected fit, and $\check{\mathcal{R}}_N^s$ and $\hat{\varepsilon}_{s,i}$ are formed in the same way from the regression of β_{st}. The term $\hat{\sigma}^2_{\Lambda_{ts}}$ is (C.6) for Λ_{ts} with its loadings estimated by plug-in from the corrected fits. In the index loading of $\Lambda^{\mathrm{reg},t}$, the plug-in replaces the expectation by the average over the cases with $\hat{Z}_i \in \mathcal{Z}'$ and $X'\mathcal{R}^t(S)$ by $\check{\mathcal{R}}_N^t(\hat{S}_i^{t,r}, X_i)$, and likewise for $\Lambda^{\mathrm{reg},s}$.

Proof of Theorem 4. Fix the t side, the s side being the same, and write $\hat{\beta}_{ts} = g_{\hat{\delta}}$ for the corrected fit and $\hat{\beta}_{ts}^{\mathrm{ls}} := g_{\hat{\delta}^{\mathrm{ls}}}$ for the least-squares fit. The decomposition

$$\begin{aligned}\hat{\mathsf{A}}^t(\hat{\beta}_{ts}) - \mathsf{A}^t(\beta_{ts}) = \mathsf{A}^t(\hat{\beta}_{ts}^{\mathrm{ls}} - \beta_{ts}) + \left(\hat{\mathsf{A}}^t(\beta_{ts}) - \mathsf{A}^t(\beta_{ts})\right) \\ + (\hat{\mathsf{A}}^t - \mathsf{A}^t)(\hat{\beta}_{ts}^{\mathrm{ls}} - \beta_{ts}) + \hat{\mathsf{A}}^t(\hat{\beta}_{ts} - \hat{\beta}_{ts}^{\mathrm{ls}})\end{aligned}$$

separates the coefficient error of the least-squares fit at $\mathsf{S}^t(z)$ (Steps 1 to 4), the evaluation at $\hat{\mathsf{S}}^t(z)$ and the product of the two errors (Step 5), and the correction (Step 6).

Step 1: the fit. Since S^t is a C^1-diffeomorphism and $\mathcal{Z}$ is compact and connected with a boundary of class $C^{2,\alpha}$ by Assumptions 4(d) and 5(b), the support $\mathrm{supp}(S_i) = \mathsf{S}^t(\mathcal{Z})$ is compact and connected with a boundary of class C^1. Any two points of $\mathrm{supp}(S_i)$ are therefore joined by a path in $\mathrm{supp}(S_i)$ whose length is at most a constant multiple of their distance. Near each point the bound holds because the boundary is locally the graph of a C^1 function, and the compactness and connectedness of $\mathrm{supp}(S_i)$ make the constant uniform. On such a set, β_{ts} extends componentwise to a function of class C^χ on $\mathbb{R}^2$, which we also denote by β_{ts}. When χ is an integer, the extension is given by Whitney's extension theorem (Whitney, 1934). When χ is not an integer, the derivatives of order $\lfloor\chi\rfloor$ of the extension must also be Hölder continuous with exponent $\chi - \lfloor\chi\rfloor$, and the version of the theorem in Stein (1970, Chapter VI, Theorem 4) provides this. Let δ^* be the coefficients of the de Boor–Fix quasi-interpolant of β_{ts} on the rectangle of Assumption 6(c), and $\beta^* := g_{\delta^*}$. By Lemma B.4, $\|\beta_{ts} - \beta^*\|_{L^\infty} \lesssim \tilde{K}_N^{-\chi}$, $\|\nabla(\beta_{ts} - \beta^*)\|_{L^\infty} \lesssim \tilde{K}_N^{1-\chi}$ and $\|\nabla\beta^*\|_{L^\infty} \lesssim 1$ on the whole rectangle, and every $\hat{S}_i$ lies in it with probability tending to one by (E.1). On $T_i = t$, $Y_i - \hat{\varphi}_i'\delta^* = \varepsilon_{t,i} + a_i - b_i$ with $a_i := X_i'(\beta_{ts} - \beta^*)(S_i)$ and $b_i := X_i'(\beta^*(\hat{S}_i) - \beta^*(S_i))$. Since L_N lies in the range of $\hat{\mathbb{Q}}_N$, the normal equations and Lemma E.1 give the identity

$$\mathsf{A}^t(\hat{\beta}_{ts}^{\mathrm{ls}} - \beta_{ts}) = \mathbb{P}_N\left[T_i^t\hat{\mathcal{R}}_N(\hat{S}_i, X_i)(\varepsilon_{t,i} + a_i - b_i)\right] - \mathbb{E}\left[T_i^t X_i'\mathcal{R}^t(S_i)a_i\right].$$

The approximation error a_i and the passthrough b_i are bounded on every cell, including

those that the boundary of $\mathrm{supp}(S_i)$ crosses, because β^* is a local approximant on the whole rectangle. In particular $\mathbb{P}_N[b_i^2] \lesssim \mathbb{P}_N[\|\hat{S}_i - S_i\|^2] = O_{\mathrm{P}}(\rho_N^2)$, by the Lipschitz bound on $\mathcal{P}$, Theorem 2(ii) and Lemma C.4.

Step 2: the outcome term. As in the argument for (7), Assumptions 1(a) and 2 make the potential treatments and outcomes independent of J_i given X_i, so the conditional law of Y_i given (T_i, X_i, J_i) depends on J_i only through Z_i. Given $\mathcal{D}_N := (T_i, W_i, J_i)_{i \le N}$, the errors $\varepsilon_{t,i}$ on $T_i = t$ are therefore independent, with mean zero by (14) and variance at most $\bar{\sigma}^2$ by Assumption 6(a), while the weights $\hat{\mathcal{R}}_N(\hat{S}_i, X_i)$ are fixed, since the first stage uses no outcome data. The term $\mathbb{P}_N[T_i^t(\hat{\mathcal{R}}_N(\hat{S}_i, X_i) - X_i'\mathcal{R}^t(S_i))\varepsilon_{t,i}]$ then has conditional mean zero and conditional variance at most $\bar{\sigma}^2 N^{-1} O_{\mathrm{P}}(\eta_N^2) = o_{\mathrm{P}}(N^{-1})$ by Lemma E.2. Hence the outcome term equals $\mathbb{P}_N[T_i^t X_i'\mathcal{R}^t(S_i)\varepsilon_{t,i}] + o_{\mathrm{P}}(N^{-1/2})$.

Step 3: the approximation term. The difference $\mathbb{P}_N[T_i^t\hat{\mathcal{R}}_N(\hat{S}_i, X_i)a_i] - \mathbb{E}[T_i^t X_i'\mathcal{R}^t(S_i)a_i]$ equals $(\mathbb{P}_N - \mathbb{P})[T_i^t X_i'\mathcal{R}^t(S_i)a_i] + \mathbb{P}_N[T_i^t(\hat{\mathcal{R}}_N(\hat{S}_i, X_i) - X_i'\mathcal{R}^t(S_i))a_i]$. The first part is $O_{\mathrm{P}}(N^{-1/2}\tilde{K}_N^{-\chi})$, and by the Cauchy–Schwarz inequality the second is $O_{\mathrm{P}}(\eta_N\tilde{K}_N^{-\chi})$. Both are $o_{\mathrm{P}}(N^{-1/2})$ under Assumption 6(d).

Step 4: the generated-regressor term. By the Cauchy–Schwarz inequality, replacing $\hat{\mathcal{R}}_N(\hat{S}_i, X_i)$ by $X_i'\mathcal{R}^t(S_i)$ in $\mathbb{P}_N[T_i^t\hat{\mathcal{R}}_N(\hat{S}_i, X_i)b_i]$ costs $O_{\mathrm{P}}(\eta_N\rho_N)$, and replacing β^* by β_{ts} in b_i costs $O_{\mathrm{P}}(\tilde{K}_N^{1-\chi}\rho_N)$. Both are $o_{\mathrm{P}}(N^{-1/2})$ under Assumptions 5(a) and 6(d). In $\mathbb{P}_N[T_i^t X_i'\mathcal{R}^t(S_i)X_i'(\beta_{ts}(\hat{S}_i) - \beta_{ts}(S_i))]$, only the cases with $S_i \in \mathsf{S}^t(\mathcal{Z}_\upsilon)$ contribute, and on these the first stage has the derivative bounds of Lemma C.5. A second-order Taylor expansion of β_{ts} and of $\boldsymbol{\gamma} \mapsto P_{\hat{\xi}}^t(\boldsymbol{\gamma}'W)$ leaves the linear term $\mathbb{P}_N[T_i^t X_i'\mathcal{R}^t(S_i)X_i'\nabla\beta_{ts}(S_i)\dot{S}_i[\hat{\theta} - \theta_o]]$ and, since $|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| = O_{\mathrm{P}}(N^{-1/2})$ by Theorem 3, a remainder of order $O_{\mathrm{P}}(\rho_N^2 + N^{-1/2}(K_N^2\rho_N + K_N^{-1}))$, which is $o_{\mathrm{P}}(N^{-1/2})$. Centering the linear term at its expectation gives $\Lambda^{\mathrm{reg},t}(\hat{\theta}) - \Lambda^{\mathrm{reg},t}(\theta_o)$ up to its remainder in Definition 3(ii). The centered part is linear in $\hat{\theta} - \theta_o$ with a fixed loading. Its index part is $O_{\mathrm{P}}(N^{-1/2})|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| = O_{\mathrm{P}}(N^{-1})$, and its propensity part is bounded by the supremum of an empirical process over the sieve ball of Definition 3(ii), which Lemma C.4 bounds by $O_{\mathrm{P}}(K_N\rho_N\sqrt{\log N/N} + K_N^3\rho_N \log N/N) = o_{\mathrm{P}}(N^{-1/2})$. The supremum covers the dependence of $\hat{\theta}$ on each case's own treatment and assignment. Hence the generated-regressor term equals $-(\Lambda^{\mathrm{reg},t}(\hat{\theta}) - \Lambda^{\mathrm{reg},t}(\theta_o)) + o_{\mathrm{P}}(N^{-1/2})$.

Step 5: the evaluation and product terms. The evaluation term is $\Lambda^{\mathrm{eval},t}(\hat{\theta}) - \Lambda^{\mathrm{eval},t}(\theta_o)$. For the product term, the change of variables $\zeta = \hat{\mathsf{S}}^t(z)$ is valid with probability tending to one, since $\|D(P_{\hat{\xi}} - P_o)\|_{L^\infty} = O_{\mathrm{P}}(K_N^2\rho_N) = o_{\mathrm{P}}(1)$ near $\mathcal{Z}_\upsilon$ by Lemma C.5. It writes the product term as the integral of $x_0'\partial_1(p(\hat{\beta}_{ts}^{\mathrm{ls}} - \beta_{ts}))$ against the difference of the weights υ^t formed from $\hat{\mathsf{S}}^t$ and from S^t. Since W_i is bounded, the Lipschitz bound on $\mathcal{P}$ and Theorem 2(i)

give $\|P_{\hat{\xi}} - P_o\|_{L^2(p_Z)} \leq \mathsf{L}(\hat{\theta}, \theta_o) + C|||\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o||| = O_{\mathrm{P}}(\rho_N)$. The inverse inequality on the first-stage sieve, with the approximation error $K_N^{1-k_o} \leq K_N\rho_N$ of the derivative, then gives $\|D(P_{\hat{\xi}} - P_o)\|_{L^2} = O_{\mathrm{P}}(K_N\rho_N)$ near $\mathcal{Z}_v$, and the difference of the weights has L^2 norm $O_{\mathrm{P}}(K_N\rho_N)$. The decomposition of Step 1, with the conditional variance of Step 2, bounds the empirical norm of the fit error by $O_{\mathrm{P}}(\tilde{K}_N/\sqrt{N} + \tilde{K}_N^{-\chi} + \rho_N)$ without any condition on the cells outside $\mathcal{S}$. The inner block of Lemma E.2 and the inverse inequality on the cells of $\mathcal{S}_N$ then give $\|\partial_1(p(\hat{\beta}_{ts}^{\mathrm{ls}} - \beta_{ts}))\|_{L^2} = O_{\mathrm{P}}(\tilde{K}_N^2/\sqrt{N} + \tilde{K}_N^{1-\chi} + \tilde{K}_N\rho_N)$ near $\mathsf{S}^t(\mathcal{Z}_v)$. The product term is therefore $O_{\mathrm{P}}(K_N\rho_N(\tilde{K}_N^2/\sqrt{N} + \tilde{K}_N^{1-\chi} + \tilde{K}_N\rho_N)) = o_{\mathrm{P}}(N^{-1/2})$ under Assumptions 5(a) and 6(d).

Step 6: the correction. The vector $\ell := \hat{L}_N - L_N$ vanishes outside $\mathcal{I} \setminus \mathcal{I}_\partial$ with probability tending to one, and $\ell' c$ is the integral of $x_0'\partial_1(pg_c)$ against the difference of the weights of Step 5. The bounds of Step 5 and the inverse inequality give $\ell'(\mathbb{Q}^{\mathrm{in}})^{-1}\ell = O_{\mathrm{P}}(\tilde{K}_N^2\nu_N^2)$, and $\ell'\hat{\mathbb{Q}}_N^{+}\ell \leq \ell'(\hat{\mathbb{Q}}_N^{\mathrm{in}})^{-1}\ell$ then gives

$$\mathbb{P}_N\left[T_i^t\left(\check{\mathcal{R}}_N^t(\hat{S}_i, X_i) - \hat{\mathcal{R}}_N(\hat{S}_i, X_i)\right)^2\right] = \ell'\hat{\mathbb{Q}}_N^{+}\ell = O_{\mathrm{P}}(\tilde{K}_N^2\nu_N^2). \tag{E.10}$$

For N large, the cases with $\hat{Z}_i \in \mathcal{Z}'$ have $Z_i \in \mathcal{Z}_N$, and by (E.1) their regressors $\hat{S}_i$ lie in cells of $\mathcal{S}_N$ that activate only basis functions in $\mathcal{I} \setminus \mathcal{I}_\partial$. On these cells, the inner block bounds the L^2 norm of a spline by its empirical norm, and the inverse inequality bounds its gradient by $C\tilde{K}_N$ times its L^2 norm. Since every null vector of $\hat{\mathbb{Q}}_N$ vanishes on $\mathcal{I}$, $\hat{\Sigma}_v$ annihilates the null space of $\hat{\mathbb{Q}}_N$. With $\hat{\mathcal{M}}_1 := \min\{\hat{\mathcal{M}}, \mathbf{I}\}$ and $\hat{q} := \hat{\mathbb{Q}}_N^{+}\hat{L}_N$, the Cauchy–Schwarz inequality in the proof of Lemma D.6 then gives

$$\left|\hat{\mathsf{A}}^t(\hat{\beta}_{ts} - \hat{\beta}_{ts}^{\mathrm{ls}})\right| = \left|\hat{L}_N'\hat{\mathbb{Q}}_N^{-1/2}\hat{\mathcal{M}}_1\hat{\mathbb{Q}}_N^{1/2}\hat{\delta}^{\mathrm{ls}}\right| \leq \left(\hat{q}'\hat{\Sigma}_v\hat{q}\right)^{1/2}\left(\hat{\delta}^{\mathrm{ls}\prime}\hat{\Sigma}_v\hat{\delta}^{\mathrm{ls}}\right)^{1/2},$$

and in the same way $\mathbb{P}_N[T_i^t(X_i'(\hat{\beta}_{ts} - \hat{\beta}_{ts}^{\mathrm{ls}})(\hat{S}_i))^2] \leq \hat{\delta}^{\mathrm{ls}\prime}\hat{\Sigma}_v\hat{\delta}^{\mathrm{ls}}$. By the form of $\hat{\Sigma}_v$ in that proof, $c'\hat{\Sigma}_v c \leq \omega_N\mathbb{P}_N[T_i^t\mathbb{1}\{\hat{Z}_i \in \mathcal{Z}'\}\|\nabla_\zeta(X_i'g_c)(\hat{S}_i)\|^2]$, where $\omega_N = O_{\mathrm{P}}(K_N^2/N)$ by Lemma D.5 and Assumption 6(d). For $c = \hat{q}$, $X_i'g_{\hat{q}}(\hat{S}_i) = \check{\mathcal{R}}_N^t(\hat{S}_i, X_i)$, and Lemma E.2 with (E.10) bounds the empirical norm of its difference from the quasi-interpolant of $X'\mathcal{R}^t$ by $O_{\mathrm{P}}(\eta_N + \tilde{K}_N\nu_N)$. The gradient of that difference is therefore $O_{\mathrm{P}}(\tilde{K}_N\eta_N + \tilde{K}_N^2\nu_N) = o_{\mathrm{P}}(1)$ in empirical norm on these cases, and the quasi-interpolant has a bounded gradient. For $c = \hat{\delta}^{\mathrm{ls}}$, the bound of Step 5 on the fit error gives the gradient bound $O_{\mathrm{P}}(\tilde{K}_N^2/\sqrt{N} + \tilde{K}_N^{1-\chi} + \tilde{K}_N\rho_N) = o_{\mathrm{P}}(1)$ in the same way, and $\nabla\beta^*$ is bounded. Both quadratic forms are therefore $O_{\mathrm{P}}(\omega_N)$, and the correction is $O_{\mathrm{P}}(K_N^2/N) = o_{\mathrm{P}}(N^{-1/2})$ under Assumption 5(a).

Step 7: the limit. Steps 1 to 6 and the same argument for the s side give

$$\bar{\tau}_{ts} - \bar{\tau}_{ts,o} = \mathbb{P}_N \left[T_i^t X_i' \mathcal{R}^t(S_i^{t,r}) \varepsilon_{t,i} - T_i^s X_i' \mathcal{R}^s(S_i^{s,r}) \varepsilon_{s,i} \right] + \Lambda_{ts}(\hat{\theta}) - \Lambda_{ts}(\theta_o) + o_{\mathrm{P}}(N^{-1/2}).$$

Theorem C.1 applied to Λ_{ts} gives the representation of the theorem. The outcome terms vanish on disjoint events, and each is uncorrelated with $\tilde{\psi}_{\Lambda_{ts}}(T_i, W_i)$ since it has mean zero given (T_i, W_i, J_i) by Step 2. The Lindeberg–Lévy central limit theorem then gives the limit with variance σ_{ts}^2.

Step 8: the variance estimator. On the cells of Step 6, the inverse inequality also bounds the supremum of a spline on a cell by $C\tilde{K}_N$ times its L^2 norm there. Lemma E.2, (E.10) and the quasi-interpolant of $\mathcal{R}^t$ therefore give

$$\max_{i:\hat{Z}_i \in \mathcal{Z}'} \left| \check{\mathcal{R}}_N^t(\hat{S}_i, X_i) - X_i' \mathcal{R}^t(S_i) \right| = O_{\mathrm{P}}(\tilde{K}_N \eta_N + \tilde{K}_N^2 \nu_N) = o_{\mathrm{P}}(1),$$

and the decomposition of Step 1 with the bound of Step 6 on the correction gives $\max_{i:\hat{Z}_i \in \mathcal{Z}'} |\hat{\varepsilon}_{t,i} - \varepsilon_{t,i}| = o_{\mathrm{P}}(1)$ in the same way. Since $\mathcal{R}^t(S_i) \neq 0$ requires $Z_i \in \mathcal{Z}_\upsilon \subset \operatorname{Int} \mathcal{Z}'$, the bound on $\max_{i \le N} \|\hat{Z}_i - Z_i\|$ gives $\mathbb{1}\{\hat{Z}_i \in \mathcal{Z}'\} \mathcal{R}^t(S_i) = \mathcal{R}^t(S_i)$ for every i with probability tending to one. The two maximal bounds, $\mathbb{P}_N[T_i^t \varepsilon_{t,i}^2] = O_{\mathrm{P}}(1)$ and the law of large numbers then give the consistency of the outcome parts of (E.9). The first-stage part is consistent by Proposition C.1 under (C.7), since the plug-in loadings converge. The propensity loadings converge in $L^2(\mathcal{Z})$, because the first two derivatives in p of the corrected fit converge in L^2 near $\mathsf{S}^t(\mathcal{Z}_\upsilon)$ by the inverse inequality with the bounds of Steps 5 and 6, using $\tilde{K}_N^3 N^{-1/2} \to 0$ for the second derivative, and the second derivatives of $P_{\hat{\xi}}$ converge in L^2 near $\mathcal{Z}_\upsilon$ at the rate $K_N^2 \rho_N$. The index loading of $\Lambda^{\mathrm{reg},t}$ converges by the first maximal bound and the convergence of the gradient of the corrected fit in empirical norm on the cases with $\hat{Z}_i \in \mathcal{Z}'$. □

F Additional Results and Discussion

F.1 Choice-Theoretic Foundation of the Assessment Model

The additive structure of (2) makes each examiner index specific to its own treatment. Raising z^1 with z^2 held fixed raises the assessment of treatment 1 alone. It moves cases into 1 from each of the other two treatments and moves no case between 0 and 2, and likewise for z^2. This is the unordered partial monotonicity of Mountjoy (2022), stated for

the pair of examiner indices, and it is the property whose absence makes two-stage least squares with a leniency instrument mix the responses at distinct margins. The converse is due to Loeser (2024). Under technical conditions, which include the absence of cases unresponsive to the indices, an assignment rule satisfying unordered partial monotonicity in the examiner indices is the maximization of assessments, each increasing in its own index. One further condition, which he states in three equivalent ways, as an identical-compliers condition, a no-defiers condition, or an additive random utility model, makes each assessment the sum of a common increasing function of its index and a case-specific term, and relabeling each index by that function gives (2). Restricted to the cases that switch between 1 and 2, the no-defiers condition is the monotonicity condition of Imbens and Angrist (1994). The economic content of the model is therefore unordered partial monotonicity together with one of these conditions, and the linearity of the examiner indices in X_i is the econometric restriction imposed on top of it. Treatment specificity does not amount to margin specificity. Raising z^1 moves cases across both margins involving treatment 1, and since any difference of assessments is determined by the other two, no shifter moves cases across a single margin alone. Under the additive form, case i assesses t and s equally under z when $z^t - z^s = U_i^t - U_i^s$, a single-index crossing of the relative index that the identification of margin-specific effects in Section 2.2 uses.

F.2 Conditional Randomization and the Augmented Specification

In the empirical setting of Section 6.1, examiner randomization holds conditional on the court and the time of disposition rather than unconditionally. Let A_i collect the circuit and disposition-time indicators of case i, so that the assignment J_i is independent of the unobservables conditional on A_i. This section describes how both stages accommodate the conditioning variables.

In the first stage, one could in principle assign examiner-specific coefficients to A_i, but we do not. Instead, A_i enters the choice index additively with examiner-invariant coefficients, with one exception: year effects interact with X_i through $X_i'\gamma_{\text{year}(i)}$, which accounts for yearly time trends in the assessments. Circuit-specific interactions with X_i require no separate terms, as they are absorbed by the finer examiner-specific interactions $X_i'\gamma_j^t$ because each examiner sits in a single circuit. The examiner-specific coefficients are therefore identified from variation within circuit and time cells, consistent with the

conditional assignment.

The second stage should also control for A_i for the validity of causal inference: the constructed regressors $S_i^{t,r}$ vary within the assignment strata, and omitting A_i would reintroduce the between-strata confounding that the design excludes. Controlling for A_i fully flexibly is nevertheless prohibitive by the curse of dimensionality. Analogously to the first stage, our primary specification enters A_i linearly along with year-varying interactions with X_i.

The time fixed effects aid first-stage identification rather than hindering it. Since the support of X_i remains almost constant except for some early years, the time effects supply a separate source of variation that spreads each examiner's index support across periods, enlarging the overlap between examiners and helping to satisfy the connectivity condition of Assumption 3. The identification analysis of Section 3 carries over to this specification unchanged. Because the common time terms enter additively, each condition of Assumption 3 applies with examiner j's index support taken as that of the full index Z_i^t of Section 6.2 rather than of $X_i'\gamma_j^t$ alone, which the time terms only enlarge.

The specification changes the first-stage estimation in three respects. Besides the examiner blocks, the parameter carries a common block of the disposition-time effects and the year-specific slopes on X_i, and the likelihood is maximized over the enlarged parameter by the blockwise ascent of Section G.1 with the common terms forming one further block.[13] The normalization of Section 3.1 extends to that block. Each family of common terms carries an additive location that trades off against the examiner intercepts, so the location restriction centers the intercepts across examiners and each family of time effects within itself, on each axis, and the scale restriction $|||\boldsymbol{\gamma}||| = 1$ is read on the examiner coefficients and the common terms together; for the details of the normalization, see Section G.1. The propensity is evaluated at the full index rather than at $X_i'\gamma_{J_i}$ alone, so the knot box and the constraint grid of Section G.1 are laid over the support of that index, and the constructed regressors of the second stage are formed from it as well.

F.3 Discussion of the Identification Conditions

This section discusses the conditions of Assumption 3 one by one, the separable propensity that Assumption 3(c) excludes, and how each condition can be assessed from the observed propensity function.

[13] The circuit indicators are absorbed by the examiner intercepts, so they do not form a separate block, unlike in the second stage where the circuit indicators are not absorbed and explicitly controlled for.

Were the nonseparability Assumption 3(c) violated for some examiner j and directions v_1, v_2, the propensity function would decouple on that examiner's index support. Moving z along v_2 would leave $v_1' P_o(z)$ unchanged, and moving z along v_1 would leave $v_2' P_o(z)$ unchanged. On any convex open subset of the support, this decoupling amounts to P_o splitting into two univariate monotone functions, each of a single linear combination of z:

$$P_o(z) = g_1(\tilde{v}_1' z)\tilde{v}_1 + g_2(\tilde{v}_2' z)\tilde{v}_2, \tag{F.1}$$

where $g_1(\cdot), g_2(\cdot)$ are strictly increasing functions and $(\tilde{v}_1, \tilde{v}_2)$ is the dual basis of (v_1, v_2), i.e., the columns of $((v_1, v_2)^{-1})'$. Geometrically, this propensity function maps each parallelogram spanned by v_1, v_2 onto a parallelogram spanned by $\tilde{v}_1, \tilde{v}_2$. Such separability is a degenerate, knife-edge property rather than a common feature. It requires the propensity function to decouple along (v_1, v_2) throughout the support, and a generic $P(\cdot)$ breaks it. For instance, Assumption 3(c) follows if the index support is $\mathbb{R}^2$. This is because a separable propensity function must map $\mathbb{R}^2$ onto the interior of a parallelogram, whereas the image of $\mathbb{R}^2$ under generic P is the interior of the probability simplex.

The examiner connectivity Assumption 3(e) does not require that every pair of examiners be comparable, since a chain of pairwise-adjacent examiners suffices. Nor does it demand a large number of examiners. Two examiners satisfy it as long as their propensity supports overlap accordingly. What it rules out is a set of examiners that splits into clusters with mutually disjoint behavior, in which case each cluster would be identified only on its own scale and location, and propensities would not be comparable across clusters. The condition is mild under complete random assignment because every examiner faces the same covariate distribution. Adjacency fails only when two examiners' coefficients map the common support of X into essentially disjoint regions of the index space, an extreme form of behavioral heterogeneity. More examiners spanning a denser range of behavior make the examiner-adjacency graph easier to connect.

The conditions of Assumption 3 are conditions on observable objects. Assumption 3(a) restricts the observed covariate distribution. Assumption 3(b) is equivalent to the Jacobian of the observed propensity function $\pi_j(x) = P_o(\gamma_{oj}' x)$ with respect to x_a attaining rank two at an interior covariate value, since this Jacobian equals $DP_o(\gamma_{oj}' x)\gamma_{oj,a}'$ and DP_o is everywhere nonsingular. Assumption 3(c) can be assessed through the linear system in γ^{-T} that the symmetry of the propensity Jacobian yields in Section 3.2. The coefficients of this system involve only the derivatives of π_j, and a violation of Assumption 3(c)

manifests itself as a solution space of dimension greater than one. Assumptions 3(d) and (e) are statements about the supports of π_j across slices and examiners. Each condition can therefore be assessed from an estimate of π_j and, for conditions (b) and (c), of its derivatives. The first stage supplies the estimate $\hat{\pi}_j(x) = \hat{P}(\hat{\gamma}_j' x)$, and its convergence to π_j does not presuppose identification. The function π_j is the conditional probability of the treatments given the examiner and the covariates, and the rate $\mathsf{L}(\hat{\theta}, \theta_o) = O_{\mathrm{P}}(\rho_N)$ of Theorem 2(i), which is the L^2 distance between $\hat{\pi}$ and π, is proved without Assumption 3 (Proposition B.1 in the Appendix). Verification in finite samples requires estimates of the derivatives of π_j, and we leave the construction of formal tests for future work.

F.4 Remarks on the First-Stage Asymptotics

The bound of Proposition 2 rests on a projection argument. Any $\hat{\boldsymbol{\gamma}} \in \Gamma$ satisfies $\boldsymbol{e}'\hat{\boldsymbol{\gamma}}^1 = \boldsymbol{e}'\hat{\boldsymbol{\gamma}}^2 = 0$ and $\langle \boldsymbol{\gamma}_o, \hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o \rangle = -\frac{1}{2}\|\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o\|^2$. $\sqrt{N}(\mathrm{vec}(\hat{\boldsymbol{\gamma}}) - \mathrm{vec}(\boldsymbol{\gamma}_o))$ therefore has vanishing intercept-mean components and radial component $o_P(1)$, the latter because regularity makes $\sqrt{N}(\mathrm{vec}(\hat{\boldsymbol{\gamma}}) - \mathrm{vec}(\boldsymbol{\gamma}_o))$ bounded in probability while the radial term is quadratic in $\hat{\boldsymbol{\gamma}} - \boldsymbol{\gamma}_o$. Every regular influence function therefore takes values in $\mathcal{T}_{\boldsymbol{\gamma}_o}$, and the projection $\Pi_{\boldsymbol{\gamma}_o}$ is common to all estimators. Efficiency is therefore decided on $\mathcal{T}_{\boldsymbol{\gamma}_o}$. A regular influence function ψ must be a gradient: $\mathbb{E}[\psi b] = 0$ for b in the nuisance tangent space, and $\mathbb{E}[\psi s_{\boldsymbol{\gamma}}'] = \Pi_{\boldsymbol{\gamma}_o}$. The canonical gradient, the efficient influence function, is $\tilde{\psi} = (I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+ s^{\mathrm{eff}}_{\boldsymbol{\gamma}}$, with variance $(I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+$. Any other gradient differs from $\tilde{\psi}$ by a term orthogonal to both $s_{\boldsymbol{\gamma}}$ and the nuisance tangent, hence to $\tilde{\psi}$, so

$$\mathrm{Var}(\psi) = (I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+ + \mathrm{Var}(\psi - \tilde{\psi}) \geq (I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^+.$$

The two projections in $s^{\mathrm{eff}}_{\boldsymbol{\gamma}} = \Pi_{\boldsymbol{\gamma}_o}(s_{\boldsymbol{\gamma}} - \Pi_{\mathcal{T}} s_{\boldsymbol{\gamma}})$ are exactly these. The residual off the nuisance tangent removes the part of the index score explained by the nuisance, and $\Pi_{\boldsymbol{\gamma}_o}$ restricts to the identified directions.

The constraints defining $\Xi(K_N)$ do not bind in the limit. Assumption 4(b) places $P_o(\cdot)$ in the interior of $\mathcal{P}$, and conditions (i), (ii) and (iv) of Section 3.1 therefore hold at $P_o(\cdot)$ with a margin uniform over $\bar{\mathcal{Z}}$. Any propensity whose values and Jacobian are uniformly close to $P_o(\cdot)$ and $DP_o(\cdot)$ on a region interior to $\mathcal{Z}$ satisfies those conditions with slack there, and $P_{\hat{\xi}}(\cdot)$ is eventually such a propensity. Fix a compact set S in the interior of $\mathcal{Z}$. With probability tending to one, every one of those inequalities is then strict at $P_{\hat{\xi}}(\cdot)$ throughout S, and no constraint anchored on S is active. The restriction

to $\mathcal{Z}$ is what makes this available, since the knot box reaches beyond the index support and the likelihood does not depend on the fitted propensity outside it. Where the fit is unrestrained by the data it is the conditions that hold it in place, and this is also the region on which $P_o(\cdot)$ is not identified.

Remark 2 (Per-examiner precision). The parametric rate in Theorem 3 should be read examiner by examiner. Only the cases assigned to examiner j carry information about γ_j, so the corresponding block of $I^{\text{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}}$ is proportional to the assignment probability λ_j, and the matching block of the asymptotic covariance carries the factor λ_j^{-1}, up to the dependence across examiners induced by the common link. Examiner j's coefficients are therefore estimated at the effective rate $(N\lambda_j)^{-1/2} \asymp N_j^{-1/2}$, where N_j is the examiner's own caseload. This concerns the examiner-specific blocks alone. Components pooled across examiners, such as the common time effects or between-examiner aggregates of the coefficients, draw on the full sample and attain the $N^{-1/2}$ rate with constants free of any single λ_j. The same holds when the estimator is applied outside the examiner design, to cross-sectional data with coefficients common to all observations, where every component is pooled. This is the formal counterpart of the caseload threshold in Section H, which excludes judges whose N_j is too small for their coefficients to be estimated with adequate precision.

F.5 Non-Necessity of Large Support of the Index

Large support of Z_i is not strictly required for the identification of $\boldsymbol{\gamma}_o$. The argument of Section 3 recovers $\boldsymbol{\gamma}_o$ from derivatives of the propensity function at interior points of $\mathcal{Z}$ and invokes no variation at the extremes of the support. Linearity of the examiner indices is what permits this. Case characteristics enter the propensity only through $\gamma_{oj}' X_i$, and finitely many coefficients therefore fix the examiner indices at every value of X_i. Variation confined to a bounded set is enough to recover them.

Compactness matters instead for f. The data reveal f only through the integrals in (3), with z ranging over $\mathcal{Z}$. Whenever a region of values of u leads to the same assigned treatment at every $z \in \mathcal{Z}$, redistributing the mass of f within that region alters no propensity, and any two densities that differ only by such a redistribution are observationally equivalent. The tails are the leading case. A case with sufficiently high values of both U_i^1 and U_i^2 is assigned to treatment 0 regardless of X_i and J_i, and a case whose U_i^1 is sufficiently low, both in level and relative to U_i^2, is assigned to treatment

1, and likewise for treatment 2. The identified nuisance is therefore $P_o(\cdot)$ on $\mathcal{Z}$, and the first-stage estimand is the pair $(\boldsymbol{\gamma}_o, P_o(\cdot))$ throughout. Were the index support unbounded instead, the criterion could be localized by a trimming indicator $\mathbb{1}\{Z_i \in \tilde{\mathcal{Z}}_N\}$ over slowly expanding sets or by a weight function, as in the trimming devices of Klein and Spady (1993) and Ichimura (1993). The localization would reach beyond the criterion, since the knot box would expand with $\tilde{\mathcal{Z}}_N$ and the overlap constant κ in the definition of the function class $\mathcal{P}$ would become a sequence tending to zero, as every propensity falls below any fixed threshold far out in the index space. The convergence rates would then carry additional factors from the localization, and we do not pursue this extension here.

F.6 The Margin at a Single Value of the Unobserved Components

Proposition 1 identifies the mean potential outcome on the (t, s) margin, a half-line in the space of the unobserved components. This subsection identifies the mean potential outcome at the endpoint of that half-line, the value of U_i at which the three assessments are equal, from the object of Proposition 1 by one further derivative.

Fix $t \neq s$ in $\{0, 1, 2\}$ with remaining treatment r. Define

$$\mu_{tsr}(z, x) = \mathbb{E}[Y_i(t) \,|\, X_i = x, U_i = (z^1, z^2)],$$

the mean potential outcome under t for the cases that assess all three treatments equally, the parameter of Lee and Salanié (2018). Let $\tilde{z} = (z^{t,r}, z^{s,r})$ and $\tilde{P}^t$ be the reparametrization of Step 1 of the proof of Proposition 1, in which r is the base treatment, and define

$$\phi_{ts}(z) = -\partial_{\tilde{z}^s} \tilde{P}^t(\tilde{z}),$$

the rate at which the t-treatment propensity falls as the s-index rises with the t-index held fixed, both relative to the r-index. When $s \neq 0$, it equals $-\partial_{z^s} P^t(z)$. When $s = 0$, it equals $\partial_{z^1} P^t(z) + \partial_{z^2} P^t(z)$ by the chain rule.

The following proposition identifies $\mu_{tsr}(z, x)$ from a "third"-stage regression of the margin-specific mean potential outcome $\mu_{ts}(Z_i, X_i)$, obtained from the second stage, on $(\phi_{ts}(Z_i), Z_i^{t,s}, X_i)$ and its derivative in $\phi_{ts}(Z_i, X_i)$, which are first-stage plug-in quantities.

Proposition F.1. *Suppose $u \mapsto \mathbb{E}[Y_i(t) \,|\, X_i = x, U_i = u]$ is bounded and continuous for every $x \in \operatorname{supp}(X_i)$. Then the map $z \mapsto (\phi_{ts}(z), z^{t,s})$ is one-to-one, and $\mu_{ts}(z, x) = q_{ts}(\phi_{ts}(z), z^{t,s}, x)$ defines a function q_{ts} on the range of that map. If $q_{ts}(\phi, b, x)$*

is continuously differentiable in ϕ, then for every $(z,x) \in \operatorname{supp}(Z_i, X_i)$,

$$\mu_{tsr}(z,x) = \left[q_{ts}(\phi,b,x) + \phi \partial_\phi q_{ts}(\phi,b,x)\right]_{\phi=\phi_{ts}(z),b=z^{t,s}}.$$

Proof of Proposition F.1. It is convenient to work in the coordinates of Step 1 in the proof of Proposition 1, in which the assessments of r, t and s are 0, $\tilde{z}^t - \tilde{U}_i^t$ and $\tilde{z}^s - \tilde{U}_i^s$, where $(\tilde{U}_i^t, \tilde{U}_i^s)$ is an invertible linear transformation of U_i and therefore has a continuous, everywhere positive density $\tilde{f}(\tilde{u}^t, \tilde{u}^s)$ satisfying the same tail bound. Write $\tilde{h}_t(\tilde{u}, x)$ for $\mathbb{E}[Y_i(t) \,|\, X_i = x, U_i = u]$ expressed in these coordinates, which is bounded and continuous by hypothesis, and $b = \tilde{z}^t - \tilde{z}^s = z^{t,s}$. The (t,s) margin event is $\{\tilde{z}^t - \tilde{U}_i^t = \tilde{z}^s - \tilde{U}_i^s > 0\}$, the half-line $\{\tilde{u}^s = \tilde{u}^t - b,\ \tilde{u}^t < \tilde{z}^t\}$ in the space of $(\tilde{u}^t, \tilde{u}^s)$, and its endpoint $(\tilde{z}^t, \tilde{z}^s)$ is the point at which the three assessments vanish. In the sense fixed in Section 2.2, the conditional expectation on the margin event is the conditional expectation given $\tilde{U}_i^t - \tilde{U}_i^s = b$ and $X_i = x$, restricted to the half-line, so that

$$\mu_{ts}(z,x) = \frac{\psi(\tilde{z}^t, b, x)}{\phi(\tilde{z}^t, b)},$$

where

$$\psi(\tilde{z}^t, b, x) = \int_{-\infty}^{\tilde{z}^t} \tilde{h}_t(\tilde{u}^t, \tilde{u}^t - b, x) \tilde{f}(\tilde{u}^t, \tilde{u}^t - b) d\tilde{u}^t,$$
$$\phi(\tilde{z}^t, b) = \int_{-\infty}^{\tilde{z}^t} \tilde{f}(\tilde{u}^t, \tilde{u}^t - b) d\tilde{u}^t.$$

Both integrals are finite by the tail bound and the boundedness of $\tilde{h}_t$, and the denominator is positive. By Lemma I.2 applied to the reparametrized model, the denominator is $\phi(\tilde{z}^t, b) = -\partial_{\tilde{z}^s} \tilde{P}^t(\tilde{z}) = \phi_{ts}(z)$, so the numerator is $\psi(\tilde{z}^t, b, x) = -\partial_{\tilde{z}^s} \mathbb{E}[Y_i T_i^t \,|\, X_i = x, \tilde{Z}_i = \tilde{z}]$. Fix b and regard ψ and ϕ as functions of $\tilde{z}^t$. Their integrands are continuous in $\tilde{u}^t$, so the fundamental theorem of calculus gives

$$\partial_{\tilde{z}^t} \phi(\tilde{z}^t, b) = \tilde{f}(\tilde{z}^t, \tilde{z}^s) > 0, \qquad \partial_{\tilde{z}^t} \psi(\tilde{z}^t, b, x) = \tilde{h}_t(\tilde{z}^t, \tilde{z}^s, x) \tilde{f}(\tilde{z}^t, \tilde{z}^s),$$

where $\tilde{z}^t - b = \tilde{z}^s$ was used. The first display shows that $\tilde{z}^t \mapsto \phi(\tilde{z}^t, b)$ is strictly increasing at every b. Since $z \mapsto (\tilde{z}^t, b)$ is an invertible linear map, $z \mapsto (\phi_{ts}(z), z^{t,s})$ is one-to-one and q_{ts} is well defined. The ratio of the two derivatives is $\tilde{h}_t(\tilde{z}^t, \tilde{z}^s, x) = \mu_{tsr}(z,x)$. Since $\psi(\tilde{z}^t, b, x) = \phi(\tilde{z}^t, b) q_{ts}(\phi(\tilde{z}^t, b), b, x)$ and q_{ts} is continuously differentiable in ϕ, the chain

rule gives

$$\partial_{\tilde{z}^t}\psi(\tilde{z}^t, b, x) = \partial_{\tilde{z}^t}\phi(\tilde{z}^t, b)\left[q_{ts}(\phi, b, x) + \phi\partial_\phi q_{ts}(\phi, b, x)\right]_{\phi=\phi(\tilde{z}^t,b)},$$

and dividing by $\partial_{\tilde{z}^t}\phi(\tilde{z}^t, b) > 0$ yields the identity. □

F.7 Two-Stage versus Joint Estimation

The joint likelihood of (Y, T) given W factors as $l(y, t \,|\, w) = l(y \,|\, t, w)l(t \,|\, w)$, where $l(y \,|\, t, w)$ is the outcome factor and $l(t \,|\, w)$ is the treatment factor. The index coefficients $\boldsymbol{\gamma}$ and the link $P(\cdot)$ parametrize the treatment factor $l(t \,|\, w)$. Whether joint estimation sharpens $\boldsymbol{\gamma}$ beyond the first-stage efficiency bound in Proposition 2 relies on what is maintained for the outcome mean, and we distinguish two cases.

Suppose first that $\mathbb{E}[Y \,|\, T, X, Z]$ is left unrestricted. Then the outcome factor carries no information about $\boldsymbol{\gamma}$, for two reasons. First, the only $\boldsymbol{\gamma}$-score the outcome carries is the mean shift $\boldsymbol{\gamma}$ induces. Under a free mean this is a perturbation of the mean itself, an element of the outcome nuisance tangent. Residualizing against that tangent therefore annihilates it. Second, the treatment-factor score $s_{\boldsymbol{\gamma}} = \partial_{\boldsymbol{\gamma}} \log l(T \,|\, W)$ is orthogonal to the whole outcome nuisance tangent. Every outcome score a is mean-zero given (T, W), so $\mathbb{E}[s_{\boldsymbol{\gamma}} a] = \mathbb{E}[s_{\boldsymbol{\gamma}}\mathbb{E}[a \,|\, T, W]] = 0$, and the outcome-nuisance projection leaves $s_{\boldsymbol{\gamma}}$ intact. The efficient score for $\boldsymbol{\gamma}$ thus reduces to its first-stage form, two-stage and joint share the same influence function, and the SMLE is efficient in the joint model.

Now impose the maintained varying-coefficient model

$$\mathbb{E}[Y \,|\, T = t, X, Z] = X'\beta_{ts}(P^t(\boldsymbol{\gamma}'W), Z^{t,r}), \quad t \in \{0, 1, 2\},$$

with an error satisfying $\mathbb{E}[\varepsilon \,|\, T, X, Z] = 0$. The index now enters the outcome non-redundantly, and the restriction over-identifies $\boldsymbol{\gamma}$. Both arguments of the varying coefficients carry $\boldsymbol{\gamma}$: the focal propensity $P^t(\boldsymbol{\gamma}'W)$ and the relative index $Z^{t,r} = Z^t - Z^r$. Conditional on $T = t$, the outcome score for $\boldsymbol{\gamma}$ runs through the mean shift

$$h = \partial_{\boldsymbol{\gamma}}[X'\beta_{ts}(P^t, Z^{t,r})] = X'[\partial_P\beta_{ts}\partial_{\boldsymbol{\gamma}}P^t(\boldsymbol{\gamma}'W) + \partial_Z\beta_{ts}\partial_{\boldsymbol{\gamma}}Z^{t,r}],$$

with $\partial_Z\beta_{ts}$ the partial in the relative-index residual argument. Each channel is bilinear in (X, W). h is therefore quadratic in X. The error takes up $h - \mathbb{E}[h \,|\, T = t, X, Z]$, the part

that depends on the examiner beyond the index residual. The varying coefficients $\beta_{ts}(\cdot)$ take up the linear-in-X part of $\mathbb{E}[h \,|\, T = t, X, Z]$ and the discrete square $X_b^{\otimes 2}$ collapses to a linear function of X_b. The remainder is reached by neither and is the over-identifying content, living in the continuous quadratic block $X_a^{\otimes 2}$ and $X_a \otimes X_b$. The SMLE is efficient for $\boldsymbol{\gamma}$ within the first-stage model but relies on limited information in the full system. The joint bound for $\boldsymbol{\gamma}$ is then weakly tighter, strictly so when this block is non-degenerate. Even then, the gain is not available to the two-stage procedure itself. The second stage fits the varying coefficients by least squares, which does not use this over-identifying content. A least-squares objective would deliver it only under the exact conditional mean with homoscedastic errors, and in general realizing the gain requires the one-step correction below or full joint estimation.

This block is degenerate, and any gain vanishes, when the varying coefficients are constant in their arguments or the continuous covariates X_a are absent. Whether a gain transmits to the structural estimands $\tau_{ts}(\cdot)$ depends on how they load on this direction, and where they do not the two-stage remains efficient for them. Write $s_{\boldsymbol{\gamma}}^{\text{out}}$ for the outcome efficient score for $\boldsymbol{\gamma}$, the outcome-factor score residualized against the outcome nuisance tangent. Regardless of its form, $s_{\boldsymbol{\gamma}}^{\text{out}}$ is mean-zero given (T, W) and hence orthogonal to the first-stage efficient score $s_{\boldsymbol{\gamma}}^{\text{eff}}$. The joint efficient score is therefore $s_{\boldsymbol{\gamma}}^{\text{eff}} + s_{\boldsymbol{\gamma}}^{\text{out}}$, and the joint information is $I_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\text{eff}} + \operatorname{Var}(s_{\boldsymbol{\gamma}}^{\text{out}})$. The link P enters the outcome only as an argument of the unrestricted functions. Its outcome-factor score is therefore absorbed, and it contributes nothing beyond the first stage. Since $\hat{\boldsymbol{\gamma}}$ already zeroes the empirical first-stage score, the one-step Newton update to the first-stage estimate is

$$\hat{\boldsymbol{\gamma}}^{+} = \hat{\boldsymbol{\gamma}} + \Big(I_{\boldsymbol{\gamma}\boldsymbol{\gamma}}^{\text{eff}} + \operatorname{Var}(s_{\boldsymbol{\gamma}}^{\text{out}})\Big)^{+} \frac{1}{N} \sum_{i=1}^{N} s_{\boldsymbol{\gamma}}^{\text{out}}(Y_i, T_i, W_i),$$

renormalized to Γ, which attains the joint bound. The update is finite-dimensional. The cost lies in forming $s_{\boldsymbol{\gamma}}^{\text{out}}$, which requires the nonparametric outcome nuisances. The correction is therefore semiparametric and costs about what joint estimation costs.

The case for the sequential procedure is primarily computational efficiency. Joint estimation maximizes a single likelihood over $\boldsymbol{\gamma}$, the link $P(\cdot)$, the varying coefficients $\beta_{ts}(\cdot)$, and the error distribution at once, coupling one finite-dimensional and three infinite-dimensional objects in a nonconvex program. The two-stage procedure splits this into two smaller and separately well-posed problems, each with its own tuning and diagnostics. The SMLE recovers $(\boldsymbol{\gamma}, P)$ on its own, and the varying coefficients follow from a flexible

regression of Y on $(P^t, Z^{t,r}, X)$ within the t-treatment subgroup. Estimating $\boldsymbol{\gamma}$ from the first stage alone does not lean on the outcome's functional form. The two-stage procedure therefore remains consistent for $\boldsymbol{\gamma}$ even when the varying-coefficient model is misspecified, whereas the one-step and joint estimators, built on it, need not be. Relaxing it to an unrestricted mean leaves no over-identifying content and returns the first case, where two-stage and joint estimation coincide asymptotically.

F.8 The Attenuation Component of the Generated-Regressor Bias and Its Correction

This subsection gives the conditions under which the expansion of Section 4.2 holds and the orders of its remainders. Fix a margin pair (t, s) with remaining treatment r, and abbreviate $S_i = S_i^{t,r}(\theta_o)$ and $\hat{S}_i = S_i^{t,r}(\hat{\theta})$ from (15). Write $\varphi_i = X_i \otimes \tilde{b}_{\tilde{K}_N}(S_i)$, $\hat{\varphi}_i = X_i \otimes \tilde{b}_{\tilde{K}_N}(\hat{S}_i)$, and $v_i = \hat{\varphi}_i - \varphi_i$, as there. Let δ stack the second-stage coefficient vectors, and adopt the exact-sieve convention $Y_i = \varphi_i'\delta + \varepsilon_i$ with $\mathbb{E}[\varepsilon_i \,|\, T_i, X_i, J_i] = 0$ on the t-subsample. For expositional purposes, we take $\mathbb{E}[v_i \,|\, T_i, X_i, J_i] = 0$ so that $\mathbb{E}[T_i^t v_i v_i']$ is the conditional variance of the construction error. The departure from this convention is the own-observation component (F.2) below, which the correction leaves in place. Let $\mathbb{Q} = \mathbb{E}[T_i^t \varphi_i \varphi_i']$ be the population Gram matrix of the regressor and $\hat{\mathbb{Q}}_N = N^{-1}\sum_{i=1}^N T_i^t \hat{\varphi}_i \hat{\varphi}_i'$ the sample Gram matrix of the constructed regressor, the objects of Section D.

Isolating the attenuation component of the bias, the variance term (17), requires no additional assumptions. The matrix Σ_v is a positive semidefinite quadratic functional of the first-stage error of a definite form, the first-stage variance $\Sigma_\zeta := \mathrm{Avar}(\hat{\theta}) = \mathbb{E}[\zeta_i \zeta_i']$ sandwiched by the loadings $X_i \otimes D\tilde{b}(S_i)\dot{S}_i$, with ζ_i the influence function of the first-stage estimator for case i, obtained from its score through a linear transformation, with $\mathbb{E}[\zeta_i] = 0$. The square root of the sandwich matrix arising from this functional reduces the quadratic form to a fixed set of linear functionals Λ of $\hat{\theta}$, obtained from its columns, so it is their joint asymptotic normality that the delta-method estimator draws on. That estimator rests on the following high-level conditions.

(i) These functionals admit the asymptotically linear representation $\Lambda(\hat{\theta} - \theta_o) = N^{-1}\sum_{i=1}^N \Lambda\zeta_i + \varpi(\Lambda)$, with remainder $\varpi(\Lambda)$ satisfying $\mathbb{E}[\varpi(\Lambda)^2] = o(N^{-1}\,\mathrm{Var}(\Lambda\zeta_i))$.

(ii) Observations are independent and identically distributed across cases.

(iii) $\tilde{b}_{\tilde{K}_N}(\cdot)$ is twice continuously differentiable with bounded second derivatives on the interior region $\mathcal{S}$ of the second-stage argument S_i.

Condition (i) is an asymptotically linear representation of these functionals, from which their joint asymptotic normality follows, the analogue for the sieve maximum likelihood first stage of the central limit theorem for smooth functionals of series estimators (Newey, 1997). Conditions (ii) and (iii) are standard in our setting.

The identity (16) is exact. Its second term has mean zero, since $\hat{\varphi}_i$ is a function of the treatments, covariates, and examiner assignments alone and the first stage uses no outcomes, so the bias sits entirely in the first term, which splits into $N^{-1}\sum_{i=1}^{N} T_i^t \varphi_i v_i' \delta$ and $N^{-1}\sum_{i=1}^{N} T_i^t v_i v_i' \delta$.

Consider the first sum. Condition (iii) gives $v_i = (X_i \otimes D\tilde{b}(S_i)\dot{S}_i)(\hat{\theta} - \theta_o) + O(\|\hat{\theta} - \theta_o\|^2)$ for each i, and Condition (i) replaces $\hat{\theta} - \theta_o$ by $N^{-1}\sum_{j=1}^{N} \zeta_j$. With the columns partitioned as $\theta = (\boldsymbol{\gamma}, \xi)$, the Jacobian of the constructed regressor is

$$\dot{S}_i = \begin{pmatrix} \xi' \, D(\partial_{z^t} b_{K_N})(Z_i) \, \dot{Z}_i & \partial_{z^t} b_{K_N}(Z_i)' \\ \dot{Z}_i^t - \dot{Z}_i^r & 0 \end{pmatrix},$$

where $\dot{Z}_i = \partial Z_i(\boldsymbol{\gamma}_o)/\partial \boldsymbol{\gamma}'$. The terms with $j \neq i$ have mean zero, since case j is independent of case i by Condition (ii) and $\mathbb{E}[\zeta_j] = 0$. The terms with $j = i$ leave the own-observation component. Every factor of that component apart from $T_i^t \zeta_i$ is a function of W_i. The influence function is a linear transformation of the first-stage score $s_i = \partial_\theta \log P^{T_i}(Z_i)$, namely $\zeta_i = \Sigma_\zeta s_i$, and $\mathbb{E}[T_i^t s_i \,|\, W_i] = P^t(Z_i)\partial_\theta \log P^t(Z_i) = \partial_\theta P^t(Z_i)$. The propensity row of $\dot{S}_i$ is $\partial_\theta P^t(Z_i)'$, and the component is therefore

$$\frac{1}{N}\mathbb{E}\left[\varphi_i \left(X_i \otimes D\tilde{b}(S_i)\dot{S}_i \Sigma_\zeta \dot{S}_i' e_1\right)'\right]\delta, \tag{F.2}$$

with e_1 the first coordinate vector of $\mathbb{R}^2$. It carries the same $N^{-1}\dot{S}_i\Sigma_\zeta\dot{S}_i'$ as Σ_v and is of the same order.

The second sum is quadratic in the first-stage error, and by Conditions (i) and (iii) its expectation equals $\Sigma_v\delta$ up to errors of order $\|\hat{\theta} - \theta_o\|^3$, with Σ_v as in (17). Its contribution (18) is the variance channel of the first-stage error, and the correction (19) offsets this channel and leaves the remaining channels of the generated-regressor bias in place, so the estimator partially corrects the bias rather than removing it in full. Condition (i) enables the substitution $\mathrm{Var}(\hat{\theta}) \approx N^{-1}\Sigma_\zeta$ in the plug-in $\hat{\Sigma}_v$ of Section 4.2, whose $\hat{\Sigma}_\zeta := \widehat{\mathrm{Avar}}(\hat{\theta})$ is

the inverse of the sample information on the eigen-directions above the threshold tol_N, the estimator of Section C.2. Under the bounded second derivative of $\tilde{b}$ from Condition (iii), the consistency of $\hat{\Sigma}_\zeta$, and the convergence of $\hat{\dot{S}}_i$, $\hat{\Sigma}_v = \Sigma_v(1 + o_{\mathrm{P}}(1))$ would hold under appropriate regularity conditions. With $\hat{\dot{S}}_i$ the Jacobian evaluated at $\hat{\theta}$, the plug-in of Section 4.2 is

$$\hat{\Sigma}_v = \frac{1}{N^2} \sum_{i=1}^{N} T_i^t (X_i \otimes D\tilde{b}_{\tilde{K}_N}(\hat{S}_i^{t,r})\hat{\dot{S}}_i)\, \widehat{\mathrm{Avar}}(\hat{\theta})\, (X_i \otimes D\tilde{b}_{\tilde{K}_N}(\hat{S}_i^{t,r})\hat{\dot{S}}_i)'.$$

The population constraint $\Sigma_v \preceq \mathbb{Q}$ holds for N large because Σ_v is of order N^{-1} while $\mathbb{Q}$ is bounded below on the retained span by Lemma D.2, which is the constraint the cap in (19) imposes on the sample spectrum.

F.9 Inference for Weighted Average Effects

This section states the class of aggregate effects for which the normal approximation behind the intervals of Section 6.3 is available, and records where the reported aggregates depart from that class. Fix a margin pair (t, s) with remaining treatment r, and let $\upsilon(z, x)$ be a known nonnegative weight function on the index and the covariates with $\int \upsilon(z, x) dz dx = 1$, where x typically refers to the discrete components of X_i in practice, so that the integral unit dx sums over the covariate cells. The weighted average effect under $\upsilon(\cdot)$ and its estimator are

$$\bar{\tau}_{ts,o}(\upsilon) = \int \upsilon(z, x)\tau_{ts}(z, x) dz dx, \qquad \bar{\tau}_{ts}(\upsilon) = \int \upsilon(z, x)\hat{\tau}_{ts}(z, x) dz dx,$$

with $\hat{\tau}_{ts}(\cdot)$ the estimated effect function of Section 4.2. The weight is required to vanish outside a compact set of z on which both arguments $(P_o^t(z), z^{t,r})$ and $(P_o^s(z), z^{s,r})$ of the effect function lie in the interiors of the regions $\mathcal{S}$ of the two regressions, and the rates of Theorem D.1 hold with $\mathcal{S}^-$ the image of this set in each regression.

At a fixed point (z, x) the estimated effect function is x' times the fitted coefficient function and its propensity derivative, evaluated at the arguments $(\hat{P}^t(z), z^{t,r}(\hat{\boldsymbol{\gamma}}))$ and $(\hat{P}^s(z), z^{s,r}(\hat{\boldsymbol{\gamma}}))$, so the estimator is a linear functional of the two fitted coefficient vectors, with the basis evaluated at these arguments, and the expansion of Section G.4 applies with Λ_{ts} and Λ_{st} the derivatives of $\bar{\tau}_{ts}(\upsilon)$ in $\hat{\delta}_{ts}^{\mathrm{ls}}$ and $\hat{\delta}_{st}^{\mathrm{ls}}$, which integrate the basis and its propensity derivative, in the coordinates $(P_o^t(z), z^{t,r})$ and $(P_o^s(z), z^{s,r})$ respectively, against $\upsilon(\cdot)$. The first-stage loading Λ_θ carries both terms of that section, the coefficient channel

and the derivative through the evaluation arguments. Let $\sigma_N^2(\upsilon)$ be the variance of the influence function under these loadings and $\hat{\sigma}_N^2(\upsilon)$ its clustered sample counterpart. The variance depends on N through the sieve dimensions K_N and $\tilde{K}_N$.

The order of $\sigma_N(\upsilon)$ separates two cases, according to whether the propensity derivative in $\tau_{ts}(\cdot)$ can be removed from the loading. Write the effect as $\mu_{ts}(z,x)-\mu_{st}(z,x)$, each term a function of its own argument, $(P_o^t(z), z^{t,r})$ or $(P_o^s(z), z^{s,r})$, through the coefficient function and its propensity derivative. When $\upsilon(\cdot)$ is continuously differentiable and vanishes in a neighborhood of the boundary of the support of Z_i, each term is rewritten in its own coordinates and its propensity derivative is integrated by parts, which moves the derivative onto the weight and the Jacobian of the change of variables and leaves no boundary term.[14] Each derivative functional then equals a level functional of the coefficient function with a bounded weight, the loading Λ_{ts} is bounded in the norm of the second-stage regression, and $\sigma_N(\upsilon)$ would be of order $O(1)$ along the sieve sequence. This is the regular case, and the weighted average is a $\sqrt{N}$-estimable functional. When $\upsilon(\cdot)$ does not vanish near the boundary, or is not differentiable, as with the indicator of a subregion, the derivative term is controlled directly rather than moved onto the weight, and compared with the level term it costs a factor of order $\tilde{K}_N$ by the spline inverse inequality, so that $\sigma_N(\upsilon)$ would be of order at most $\tilde{K}_N$. This is the irregular case.

In both cases the approximation

$$\sqrt{N}\left(\bar{\tau}_{ts}(\upsilon)-\bar{\tau}_{ts,o}(\upsilon)\right)/\hat{\sigma}_N(\upsilon) \to_d N(0,1)$$

holds under three conditions, in the form of the sieve t-statistic of Chen and Pouzo (2015). First, the variance estimator is consistent in ratio, $\hat{\sigma}_N(\upsilon)/\sigma_N(\upsilon) \to 1$ in probability. In the regular case this is consistency of the clustered variance estimator, and in the irregular case the ratio form is needed because $\sigma_N(\upsilon)$ grows. Second, the bias of the estimator is small against $\sigma_N(\upsilon)/\sqrt{N}$. The bias has three sources, and the rates of Theorems 2(ii) and D.1 give their orders. The sieve approximation error of the second stage at the pseudo-true first-stage argument enters through $\int \upsilon(z,x)x'\{(\beta_{ts}^*-\beta_{ts})(s^t)+P_o^t(z)\partial_1(\beta_{ts}^*-\beta_{ts})(s^t)\}dzdx$ with $s^t=(P_o^t(z), z^{t,r})$, and its counterpart on the s side, with β_{ts}^* the population least-squares coefficient function at that argument. In the irregular case this functional is continuous in the norm $\|g\|_{L^\infty(\mathcal{S})}+\|\partial_1 g\|_{L^\infty(\mathcal{S})}$ and the term is of order $\tilde{K}_N^{-(\chi-1)}$. In the

[14] The changes of variables use the maps $z \mapsto (P_o^t(z), z^{t,r})$ and $z \mapsto (P_o^s(z), z^{s,r})$, whose Jacobians are continuously differentiable on the support of $\upsilon(\cdot)$ under Assumption 4.

regular case the integration by parts removes the derivative and the term is of order $\tilde{K}_N^{-\chi}$. The first-stage approximation error, of order $K_N^{-k_o}$ under Assumption 4, passes through the evaluation arguments and contributes $K_N^{-k_o}$ in both cases, since the derivatives of β_{ts}^* up to the second order are bounded on a quasi-uniform mesh whenever those of β_{ts} are. The terms of second order in $\hat{\theta} - \theta_o$, collected in the linear functional $\Lambda_{ts}' \hat{\mathbb{Q}}_{ts}^{+} \varsigma_{ts}$ of the remainder of Section G.4 and its counterpart, are of the order of the attenuation term of Section F.8, $\tilde{K}_N^2 \rho_N^2$, with or without the correction of (19), by Lemmas D.5 and D.6. This order is a property of the fitted coefficients and is common to the two cases. The bias condition would therefore read $\sqrt{N}(\tilde{K}_N^{-(\chi-1)} + K_N^{-k_o} + \tilde{K}_N^2 \rho_N^2) = o(\sigma_N(\upsilon))$ in the irregular case and $\sqrt{N}(\tilde{K}_N^{-\chi} + K_N^{-k_o} + \tilde{K}_N^2 \rho_N^2) \to 0$ in the regular case. Third, the cluster sums that form the influence function satisfy the Lindeberg condition, so that no single cluster contributes a nonvanishing share of the variance.

The reported aggregate $\bar{\tau}_{ts}(\mathcal{X})$ of Section 6.3 has the form $\bar{\tau}_{ts}(\hat{\upsilon}_{\mathcal{X}})$ with $\hat{\upsilon}_{\mathcal{X}}(z, x)$ the product of the taper weight $\upsilon(z)$ of that section, normalized to integrate to one, and the empirical distribution of X_i over the cases with $X_i \in \mathcal{X}$, and its estimand $\bar{\tau}_{ts,o}(\mathcal{X})$ is the same average with the coefficients at their population values and the evaluation points at their estimated values. The weight is of the regular kind, since the taper $\upsilon(\cdot)$ is twice continuously differentiable and vanishes outside the interior region $\{z \in \mathbb{R}^2 : d(z) \leq d_q\}$ of that section. It differs from the class above in one respect: it is formed from the sample and from the first-stage estimates rather than known in advance, and the estimand and the standard errors of Section G.4 hold it fixed at its estimated value, so that the first-stage loading carries the coefficient channel alone. The intervals for the reported aggregates accordingly rest on the three conditions of the regular case.

G Implementation Details

G.1 First-Stage Estimation

The first stage maximizes the likelihood jointly over the examiner coefficient blocks, the time effects, and the surplus spline coefficients, and this subsection records the implementation on which the reported fits rest. Optimization proceeds by blockwise gradient ascent with block-specific step sizes. Two devices condition the examiner blocks. First, the covariates are prewhitened examiner by examiner and the gauge normalization is imposed in the prewhitened coordinates, which makes the steps invariant to the scaling

of the covariates and is important for convergence at the empirical dimensions. Second, each examiner's gradient is preconditioned by the inverse of the outer product of that examiner's index scores at the current iterate, with a relative ridge of 10^{-4}. By the information equality the outer product of the scores estimates the information, so in a likelihood ascent the preconditioned step approximates a per-examiner Newton step. Since the outer product aggregates over that examiner's cases while the pooled gradient weights each examiner by the caseload share, the preconditioning also adjusts each examiner's effective step by the case count, and small-caseload examiners converge at the rate of the others. Preconditioning of this kind greatly helps convergence at the empirical dimensions. The examiner coefficients are initialized at zero, and the surplus coefficients are initialized by fitting the sieve to the multinomial logit propensities, a warm start that places the initial iterate inside the region where the shape restrictions hold.

The Jacobian symmetry that drives identification holds for every element of the gradient B-spline sieve and requires no enforcement. The remaining restrictions of the propensity class enter as constraints on the spline coefficients, imposed on a finite grid of index values. The constraints enforced are the monotonicity restrictions and the diagonal dominance they induce. Positive semidefiniteness of the propensity Jacobian, the weaker condition identification requires, is monitored. An iterate that leaves the constrained set is returned to it by Dykstra's alternating projection algorithm, a choice compatible with the blockwise ascent. The grid should be neither too fine nor too coarse, since a finer grid raises the cost of every projection while a coarser one leaves the restrictions unchecked between grid points. Our rule sets the constraint-grid mesh at three quarters of the knot mesh. In the empirical fits the knots are uniform with mesh 0.4 on the box $[-2.4, 4.4]^2$ and the constraint grid has mesh 0.3 on the box $[-2.4, 3.9]^2$, and we have explored alternative meshes around this rule. Throughout the implementation the complexity of the sieve is indexed by the knot mesh rather than by the number of basis functions, the two being equivalent on the fixed box. The projection enforces the constraints on the grid alone and not at the realized index values, and after convergence we audit positive semidefiniteness of the Jacobian at the estimated index of every case in the sample. At the main specification 1.27 percent of the cases violate it, with a smallest eigenvalue of -0.019 at the worst case and -0.007 on average among the violating cases, against a median largest eigenvalue of 1.36, and the violations lie away from the interior region on which the second stage is fitted. Imposing the constraints at finitely many points stands in for imposing them at every point, and how closely it does so is measurable. A monitoring grid finer than the

enforcement grid reads where nothing was imposed, reporting the slack in the eigenvalues of $DP(\cdot)$ along the ascent. This is justified because the identification argument of Section 3 uses less than $\mathcal{P}$ imposes: the gradient structure that the sieve supplies at no cost together with the positive semidefiniteness of $DP(\cdot)$ that the monotonicity constraints imply.

The scale normalization of Section 3.1 is read on the prewhitened coordinates. With the non-intercept covariates centered within each examiner's caseload, the restriction imposed in the implementation is

$$\sum_{j=1}^{J} \operatorname{tr}\left(\gamma_j'\left(\frac{1}{N_j}\sum_{i:J_i=j} X_i X_i'\right)\gamma_j\right) + \sum_{t\in\{1,2\}}\left(\|\gamma_{\text{year}}^t\|^2 + \|\alpha_{\text{month}}^t\|^2 + \|\alpha_{\text{dow}}^t\|^2\right) = \varrho^2,$$

where γ_j is the block of examiner j, N_j is that examiner's caseload, γ_{year}^t stacks the year fixed effects and the year slopes of Section 6.2, α_{month}^t and α_{dow}^t stack the month and day-of-week effects, and the norms on the common terms are Euclidean. In these units the squared norm of examiner j's slopes for treatment t is the variance of the t-treatment index across that examiner's cases, and under the location normalization the intercepts add the between-examiner dispersion of the index levels, so that ϱ^2/J is the combined variance of the two indices for the average examiner. We set $\varrho^2 = 0.1J$, so that each index has a standard deviation of roughly 0.22 when the two contribute comparably. The range and resolution of the knot box and the constraint grid are specified relative to this scale, whereas the second stage is unaffected by it, since the fitted propensities are invariant to the choice and a common rescaling of the relative index is absorbed by the second-stage knots.

No larger reparametrization than the two invariances of Section 3.1 is available, because any admissible propensity function has a symmetric Jacobian by condition (iii), which a nonscalar linear mixing of the two indices would destroy. Parametric specifications require no counterpart, since assuming a complete distribution for U_i, as the multinomial logit does, already fixes the location and scale of the errors. The reported objects, the propensities, the ranking of cases by their indices, and the second-stage functionals, do not depend on the normalization chosen. A common alternative in index models sets one designated coefficient to unity, which here means fixing one component of one examiner's coefficient vector. That normalization requires knowing in advance that the designated coefficient is nonzero, together with its sign, and the coefficients of every examiner in both treatments are then identified relative to it, diverging as its true value approaches zero. The restriction $|||\boldsymbol{\gamma}||| = 1$ requires only $\boldsymbol{\gamma}_o \neq 0$, treats all components symmetrically, and

preserves the compactness of Γ.

The reported fit runs 20,000 iterations of the blockwise ascent with no early stopping, and convergence is monitored through the score of the index block. The step sizes are 10^{-3} for the examiner blocks, 0.1 for the surplus coefficients, and 5×10^{-3} for the year slopes and for the time fixed effects, and each projection runs 200 Dykstra iterations at maximum. Each preconditioned step is projected onto the tangent space of the scale normalization at the current iterate, and the iterate is then returned to the normalization surface by rescaling, a retraction whose correction to the projected step is of second order. The logit is estimated by gradient ascent with a step size of 0.1, each examiner's gradient averaged over that examiner's cases and preconditioned by the inverse of that examiner's covariate second-moment matrix, which equalizes the convergence rates across examiners in the same way. The Monte Carlo experiments of Section 7 run 10,000 iterations with the step sizes doubled and the preconditioner ridge at 4×10^{-4}.

G.2 Estimation of the Asymptotic Variance of the Index Coefficients

Remark 3 (Estimation of asymptotic variance of $\hat{\boldsymbol{\gamma}}$). As noted in Ackerberg et al. (2012), the efficient information in the first-stage model can be estimated as if the sieve propensity were parametrically specified. Let $\hat{s}_{\boldsymbol{\gamma},i} = \partial_{\boldsymbol{\gamma}} \log l(T_i \,|\, W_i, \hat{\theta})$ and $\hat{s}_{\xi,i} = \partial_{\xi} \log l(T_i \,|\, W_i, \hat{\theta})$ denote the estimated index and nuisance scores of observation i. Let $\Pi_{\hat{\boldsymbol{\gamma}}} = \mathbf{I}_L - \mathrm{vec}(\hat{\boldsymbol{\gamma}})\mathrm{vec}(\hat{\boldsymbol{\gamma}})'$ be the projection onto the tangent space $\mathcal{T}_{\hat{\boldsymbol{\gamma}}}$. Then, under the conditions of Theorem 3, the efficient information can be estimated by

$$\hat{I}^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}} = \Pi_{\hat{\boldsymbol{\gamma}}} \left(\frac{1}{N} \sum_{i=1}^{N} \hat{s}_{\boldsymbol{\gamma},i} \hat{s}'_{\boldsymbol{\gamma},i} - \frac{1}{N} \sum_{i=1}^{N} \hat{s}_{\boldsymbol{\gamma},i} \hat{s}'_{\xi,i} \left(\frac{1}{N} \sum_{i=1}^{N} \hat{s}_{\xi,i} \hat{s}'_{\xi,i} \right)^{+} \frac{1}{N} \sum_{i=1}^{N} \hat{s}_{\xi,i} \hat{s}'_{\boldsymbol{\gamma},i} \right) \Pi_{\hat{\boldsymbol{\gamma}}}.$$

To estimate the asymptotic covariance matrix $\Sigma = (I^{\mathrm{eff}}_{\boldsymbol{\gamma}\boldsymbol{\gamma}})^{+}$, let $\hat{s}_i = (\Pi_{\hat{\boldsymbol{\gamma}}} \hat{s}'_{\boldsymbol{\gamma},i}, \hat{s}'_{\xi,i})'$ be the grand score vector of observation i, where $\Pi_{\hat{\boldsymbol{\gamma}}} \hat{s}_{\boldsymbol{\gamma},i}$ is the projected index score. Compute the outer product $\hat{s}_i \hat{s}'_i$ and take its sample average $N^{-1} \sum_{i=1}^{N} \hat{s}_i \hat{s}'_i$. The asymptotic covariance matrix can be estimated by the upper-left block of the Moore–Penrose inverse of this average

$$\hat{\Sigma} = \left(\frac{1}{N} \sum_{i=1}^{N} \hat{s}_i \hat{s}'_i \right)^{+}_{1:2\dim(W),1:2\dim(W)}.$$

One can also use the recentered version $\check{\Sigma} = (N^{-1}\sum_{i=1}^{N}\hat{s}_i\hat{s}_i' - \bar{\hat{s}}\bar{\hat{s}}')^{+}_{1:2\dim(W),1:2\dim(W)}$ with $\bar{\hat{s}} = N^{-1}\sum_{i=1}^{N}\hat{s}_i$, which may be more appropriate when the first-order conditions are not exactly enforced such as under early stopping of the optimization. Proposition C.1 in the Appendix shows that these estimators are consistent once their spectrum is appropriately regularized, so that the t and Wald statistics built from them are asymptotically standard.

G.3 Spectral Regularization

The procedure involves inverting two symmetric positive semidefinite matrices whose smallest eigenvalues are not necessarily bounded away from zero: the joint first-stage information over the index coefficients and the surplus coefficients and the second-stage Gram matrix $\hat{\mathbb{Q}}_N$.

The pseudo-inverse of the joint first-stage information yields $\widehat{\mathrm{Var}}(\hat{\theta})$ and is used for computing the standard errors for second-stage estimates and $\hat{\Sigma}_\zeta$. The pseudo-inverse of the second-stage Gram is used in three places: to compute the least-squares coefficient $\hat{\delta}^{\mathrm{ls}}$, to compute the directional derivatives of the effect functionals with respect to $\hat{\theta}$, and to compute the correction of (19). All three tend to be numerically stable, since each inherits a contraction property of the least squares projection. The second-stage Gram is therefore inverted at its pseudo-inverse with rank determined at rounding precision and no statistical threshold.

The joint first-stage information is treated differently for structural reasons. Its surplus block is essentially singular along directions carried by cells with few or no observations, and those directions are trimmed, whereas the index block is nonsingular on the tangent space by Lemma C.1, so its retained spectrum is floored rather than trimmed, at tol times the largest eigenvalue, which bounds the inverse in finite samples without removing any direction that carries information. The floor rests on the information-theoretic reading of the spectrum: an eigenvalue counts the information accumulated in its direction, and a direction whose eigenvalue stays bounded as N grows carries no accumulating information and is weakly identified.

The joint first-stage information is singular by nature: its index block is written in the coordinates of the tangent space, so the directions removed by the normalization are exact zeros. The surplus block is restricted to the directions whose eigenvalue in the surplus information exceeds tol_ξ times its largest, so that a basis coefficient the sample does not support is explicitly removed. Let n_s be the rank of the normalization plus the

number of surplus directions trimmed, and let λ_s denote the n_s-th smallest eigenvalue of the information. A direction is retained when its eigenvalue is at least $M\lambda_s$, where $M \geq 1$ is a multiplier of choice. The retained eigenvalues are then floored below at $\mathsf{tol}\lambda_1$ before inversion, with λ_1 the largest eigenvalue, which keeps the inverse bounded in finite samples without removing any retained direction. The floor carries the information-theoretic reading of the threshold. The joint information aggregates the scores over the sample, so an eigenvalue counts the information accumulated in its direction, and $\mathsf{tol} = c/N$ places the floor at the information of c observations in the most informative direction. By Lemma C.1 the efficient information $I^{\mathrm{eff}}_{\gamma\gamma}$ is nondegenerate on the tangent space $\mathcal{T}_{\gamma_o}$, so the index coefficients are strongly identified: every retained eigenvalue of the joint information is of order N, and a floor of the form c/N ceases to bind as the sample grows. In a finite sample it bounds the pseudo-inverse: since $1/(N\mathsf{tol}) = 1/c$, no direction contributes more variance than $1/c$ times the inverse of one observation's information in the most informative direction, and the value $c \approx 2.6$ used here prevents the pseudo-inverse from exploding in the empirical fits.

All reported estimates and the Monte Carlo experiments of Section 7 use $\mathsf{tol}_\xi = 10^{-4}$, the information of about 26 observations at $N = 256{,}475$, $M = 5$, and $\mathsf{tol} = 10^{-5}$.

G.4 Inference for the Reported Aggregates

This section derives the standard errors reported in Section 6.3, accounting for the outcome error and for the first-stage error carried by the generated regressors. Fix a margin pair (t, s) with remaining treatment r, and write $\varphi_{ts,i}(\theta) = ((X_i \otimes \tilde{b}_{\tilde{K}_N}(S_i^{t,r}(\theta)))', A_i')'$ for the second-stage regressor vector, which extends the regressor φ_i of Section F.8 by the additive controls A_i of the empirical specification, where $S_i^{t,r}(\theta)$ is defined in (15) and A_i does not depend on θ. The fit and its Gram matrix are $\hat{\delta}^{\mathrm{ls}}_{ts} = \hat{\mathbb{Q}}^{+}_{ts} N^{-1} \sum_{i=1}^{N} T_i^t \varphi_{ts,i}(\hat{\theta}) Y_i$ and $\hat{\mathbb{Q}}_{ts} = N^{-1} \sum_{i=1}^{N} T_i^t \varphi_{ts,i}(\hat{\theta}) \varphi_{ts,i}(\hat{\theta})'$, where δ_{ts} now stacks the varying-coefficient blocks and the control coefficients, and $\hat{\mathbb{Q}}_{ts}$ extends the sample Gram matrix $\hat{\mathbb{Q}}_N$ of Section F.8 by the coordinates of A_i. The reported quantity $\bar{\tau}_{ts}(\mathcal{X})$ of that section integrates the estimated effect function against the taper weight $v(\cdot)$ and averages it over the cases with $X_i \in \mathcal{X}$. The estimand, denoted $\bar{\tau}_{ts,o}(\mathcal{X})$, is this average with the coefficients at their population values and the evaluation points at their estimated values. The evaluation points are part of the definition of the reported quantity, and the first-stage error therefore enters only through the fitted coefficients $\hat{\delta}^{\mathrm{ls}}_{ts}$ and $\hat{\delta}^{\mathrm{ls}}_{st}$.

Write the estimating equation of the fit as $N^{-1}\sum_{i=1}^{N}\Omega_{ts,i}(\delta_{ts},\theta)=0$ with $\Omega_{ts,i}(\delta_{ts},\theta)=T_i^t\varphi_{ts,i}(\theta)(Y_i-\varphi_{ts,i}(\theta)'\delta_{ts})$. Let $\delta_{ts,o}$ be the population least-squares coefficient at θ_o and $\varepsilon_{t,i}=Y_i-\varphi_{ts,i}(\theta_o)'\delta_{ts,o}$ the projection error. Under the exact-sieve convention of Section F.8 this error is the outcome error of the t-subsample, common to the two fits on that cell, so it carries the cell index alone. Expanding around $(\delta_{ts,o},\theta_o)$ gives

$$\hat{\delta}_{ts}^{\mathrm{ls}}-\delta_{ts,o}=\hat{\mathbb{Q}}_{ts}^{+}\frac{1}{N}\sum_{i=1}^{N}T_i^t\varphi_{ts,i}\varepsilon_{t,i}+\hat{\mathbb{Q}}_{ts}^{+}M_{ts}(\hat{\theta}-\theta_o)+\hat{\mathbb{Q}}_{ts}^{+}\varsigma_{ts},$$
$$M_{ts}=\mathbb{E}\left[\frac{\partial\Omega_{ts,i}}{\partial\theta'}(\delta_{ts,o},\theta_o)\right],$$

where $\varphi_{ts,i}$ without an argument is evaluated at θ_o. The Jacobian of the estimating equation in δ_{ts} is the Gram matrix at θ_o, and it multiplies a centered sum, so replacing it by $\hat{\mathbb{Q}}_{ts}$ changes the expansion by a term of second order. The remainder ς_{ts} collects that term, the terms of second order in $\hat{\theta}-\theta_o$, among them the attenuation term of Section F.8, and the deviation of $N^{-1}\sum_{i=1}^{N}\partial\Omega_{ts,i}/\partial\theta'$ from its expectation. The reported aggregates involve ς_{ts} only through linear functionals of $\hat{\mathbb{Q}}_{ts}^{+}\varsigma_{ts}$, and it is the order of those functionals that the bias condition of Section F.9 constrains. The derivative of $\Omega_{ts,i}$ in θ has two parts, one proportional to $\varepsilon_{t,i}$ and one collecting the movement of the regressor vector through the Jacobian $\dot{S}_i$ of Section F.8. The first part has mean zero under

$$\mathbb{E}[\varepsilon_{t,i}\,|\,T_i,X_i,J_i]=0, \tag{G.1}$$

the mean independence Section F.8 already imposes on the outcome error, and Conditions (i) to (iii) of that section are maintained here. Writing $\dot{\varphi}_{ts,i}=\partial\varphi_{ts,i}(\theta)/\partial\theta'$ at θ_o, which is a function of (X_i,W_i) alone, the loading is therefore

$$M_{ts}=-\mathbb{E}\left[T_i^t\varphi_{ts,i}\delta_{ts,o}'\dot{\varphi}_{ts,i}\right].$$

Let Λ_{ts} denote the derivative of $\bar{\tau}_{ts}(\mathcal{X})$ in $\hat{\delta}_{ts}^{\mathrm{ls}}$, and let Λ_{st} be its derivative in $\hat{\delta}_{st}^{\mathrm{ls}}$, which carries a negative sign since the two fits enter the effect as a difference. Collect the first-stage loading as $\Lambda_\theta=M_{ts}'\hat{\mathbb{Q}}_{ts}^{+}\Lambda_{ts}+M_{st}'\hat{\mathbb{Q}}_{st}^{+}\Lambda_{st}$. When the estimand fixes the evaluation arguments at their population values, as for the functionals of Section F.9, Λ_θ gains a second term, the derivative of the aggregate in θ through the arguments $(P^t(z),z^{t,r})$ and $(P^s(z),z^{s,r})$ at which the fitted coefficient functions and their propensity derivatives are evaluated. Combining the expansion above with the asymptotically linear representation

of Condition (i) of Section F.8, applied to the linear functional $\Lambda_\theta'\hat{\theta}$, the influence function of the reported aggregate is

$$\psi_i = \frac{1}{N}\left(\Lambda_{ts}'\hat{\mathbb{Q}}_{ts}^{+}T_i^t\varphi_{ts,i}\varepsilon_{t,i} + \Lambda_{st}'\hat{\mathbb{Q}}_{st}^{+}T_i^s\varphi_{st,i}\varepsilon_{s,i} + \Lambda_\theta'\zeta_i\right),$$

with ζ_i the first-stage influence function of Section F.8, the sampling error of the reported aggregate being $\sum_{i=1}^N \psi_i$. The outcome part and the first-stage part are uncorrelated: the score of case i is a function of (T_i, X_i, J_i) alone, the first stage uses no outcomes, and (G.1) removes the cross moment. No covariance term therefore enters the variance.

Every loading is computed from the fit. Λ_{ts} is the coefficient vector of the linear map from $\hat{\delta}_{ts}^{\text{ls}}$ to $\bar{\tau}_{ts}(\mathcal{X})$, and $\hat{\mathbb{Q}}_{ts}^{+}M_{ts}$ is the negative of the coefficient matrix of the least-squares regression of $\hat{\varphi}_{ts,i}'\hat{\delta}_{ts}^{\text{ls}}$ on $\hat{\varphi}_{ts,i}$ within the t-treatment subsample, both evaluated at $\hat{\theta}$. The variance is estimated in clustered form at the judge-by-year level, as the sum over the clusters g of $(\sum_{i\in g}\hat{\psi}_i)^2$, where $\hat{\psi}_i$ is the influence function above with the estimates in place of the population quantities,

$$\hat{\psi}_i = \frac{1}{N}\left(T_i^t\Lambda_{ts}'\hat{\mathbb{Q}}_{ts}^{+}\hat{\varphi}_{ts,i}\hat{\varepsilon}_{t,i} + T_i^s\Lambda_{st}'\hat{\mathbb{Q}}_{st}^{+}\hat{\varphi}_{st,i}\hat{\varepsilon}_{s,i} + \hat{\Lambda}_\theta'\hat{\Sigma}_\zeta(\hat{s}_i - \bar{\hat{s}})\right),$$

with $\hat{s}_i$ the estimated first-stage scores of Remark 3 and $\hat{\Sigma}_\zeta$ as in that remark and Section F.8. Summing the outcome part and the first-stage part within a cluster before squaring allows for dependence within the blocks, including dependence between the outcome part of one case and the first-stage part of another.

The reported intervals are those of the corrected coefficients $\hat{\delta}$ of (19) rather than of $\hat{\delta}^{\text{ls}}$. Write $\hat{\mathcal{K}}_{ts} = \mathbf{I} + \hat{\mathbb{Q}}_N^{-1/2}\min\{\hat{\mathcal{M}}, 1\}\hat{\mathbb{Q}}_N^{1/2}$, with $\mathbf{I}$ the identity matrix, for the map of that display on the (t, s) fit, extended by the identity on the coordinates of A_i, which the correction leaves alone. With $\hat{\mathcal{K}}_{ts}$ held at its estimate, the reported aggregate is the same linear functional of the fitted coefficients with $\hat{\mathcal{K}}_{ts}'\Lambda_{ts}$ in place of Λ_{ts}, and likewise on the s side. The influence function and its variance estimator are therefore those above at the replaced loadings, in the first-stage channel through Λ_θ as well as in the outcome channel. The sampling error of $\hat{\mathcal{K}}_{ts}$ itself is of second order in $\hat{\theta} - \theta_o$ and is not carried.

The Gram inverses above are the pseudo-inverses of Section G.3. The calculation holds the trimming, the weight $\upsilon(\cdot)$, and the second-stage knots fixed at their estimated values, and treats the sieve approximation errors of both stages as negligible under undersmoothing.

H Additional Empirical Results and Robustness Checks

Sample Construction. Our extract spans 2000–2019 across 13 circuits and keeps 137 judges with more than 500 cases. We further drop judges with fewer than 50 cases involving female defendants, black defendants, or defendants convicted within the previous five years, so that each judge's slopes on these covariates are estimated with adequate precision. This results in a full sample of 256,475 cases disposed by 129 judges. For our varying-coefficient second stage, we restrict to cases disposed through 2012 as in HOSSvD. The cutoff allows each case the full seven-year follow-up that the longest recidivism window requires. Table 5 reports summary statistics by disposition, with rows and variable definitions matching Table I of HOSSvD. The two samples differ only in the judge screens, which cost about eight percent of their observations and leave the composition of the three disposition groups unchanged.

Table 6 reports the aggregate effects of Figure 2 in numbers.

Figure 6 repeats Figure 2 with a richer second-stage basis, cubic tensor B-splines with five basis functions per axis, which places one interior knot at the midpoint of the support of each fitted index. No estimate moves by more than 0.040, and every sign is preserved apart from the first-year estimate for new incarceration on the NC margin, which is indistinguishable from zero under either basis. The effects on new charges and on new incarceration over the cumulative seven-year window on the ND margin rise from 0.238 and 0.262 to 0.263 and 0.286. The intervals widen throughout, by between 9 and 25 percent.

Table 7 presents the sample correlations between the estimated judge-specific coefficients and the case characteristics, within circuit. The correlations are below 0.04 in absolute value throughout, which is what randomization of judges within circuit predicts, though the table is a descriptive check rather than a formal test, as it does not account for the sampling variation in the coefficient estimates.

H.1 Alternative First-Stage Specifications

This section records the two reference specifications against which the semiparametric first stage of Section 6.2 was set: the judge-fixed-effects model that HOSSvD estimate as an alternative to their two-stage least squares design, and a logit that shares the judge-specific slopes of our model but imposes the logistic distribution. The three differ in two features: whether the slopes on X_i vary across judges, and whether the distribution of

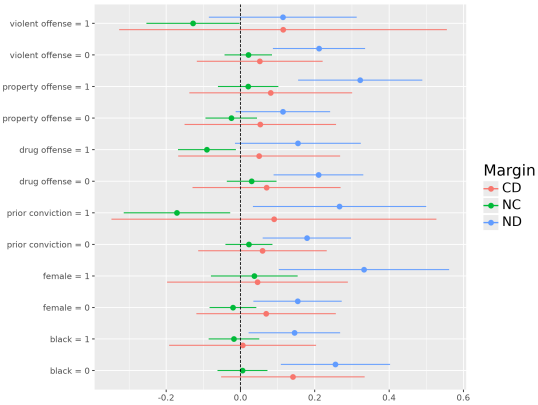
Table 5: Summary Statistics by Disposition

	Dismissed (1)	Convicted (2)	Incarcerated (3)
Offenses			
Drugs	0.36	0.33	0.29
Larceny	0.17	0.29	0.25
Assault	0.19	0.08	0.18
Fraud	0.09	0.16	0.10
Traffic	0.04	0.05	0.13
Burglary	0.06	0.07	0.08
Robbery	0.04	0.02	0.06
Sexual assault	0.03	0.02	0.03
Kidnapping	0.03	0.01	0.02
Murder	0.01	0.00	0.01
Defendant characteristics			
Black	0.57	0.51	0.60
Female	0.22	0.32	0.16
% of people in ZIP earning < $25K	0.46	0.45	0.46
Has misdemeanor	0.06	0.09	0.07
Prior conviction within five years	0.14	0.09	0.20
Sentencing			
Incarceration length	0.00	0.00	26.97
Probation length	0.00	32.08	39.86
Median incar. length	0	0	12
Median prob. length	0	12	12
Percent of sample	16	30	55
Observations	26,313	50,447	92,401

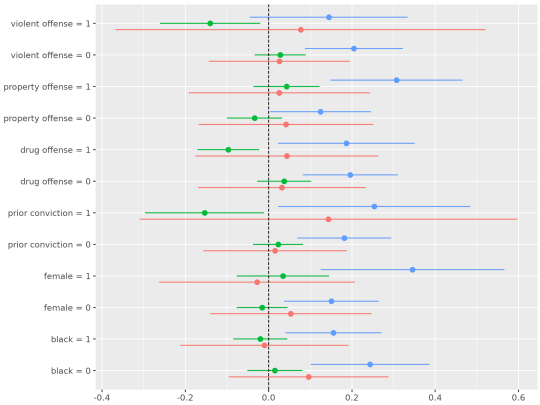
Notes. This table reports means, and medians where indicated, of case characteristics by disposition for the outcome sample of cases disposed through 2012, before the index trim described in the text. The columns correspond to cases ending in dismissal, noncarceral conviction, and incarceration. The rows and variable definitions follow Table I of HOSSvD. Incarceration and probation lengths are means in months, with probation top-coded at 20 years.

Figure 5: Cell Average Effects by Margin: New Incarceration and Years 2 to 4

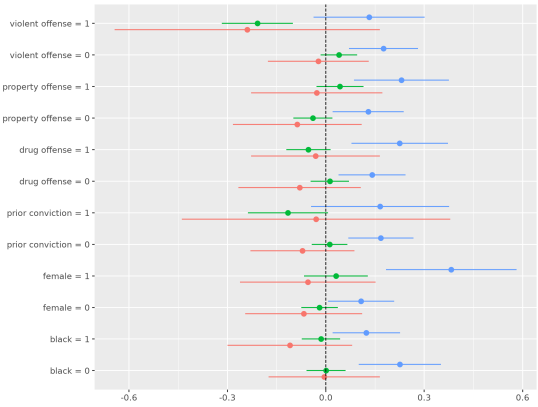
(a) New felony charge, years 2–4



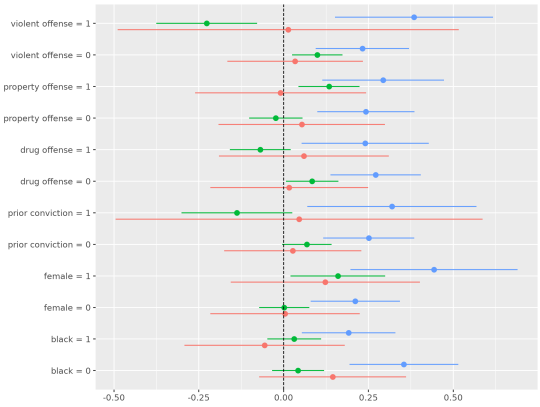
(b) New felony conviction, years 2–4



(c) New incarceration, years 2–4



(d) New incarceration, years 1–7



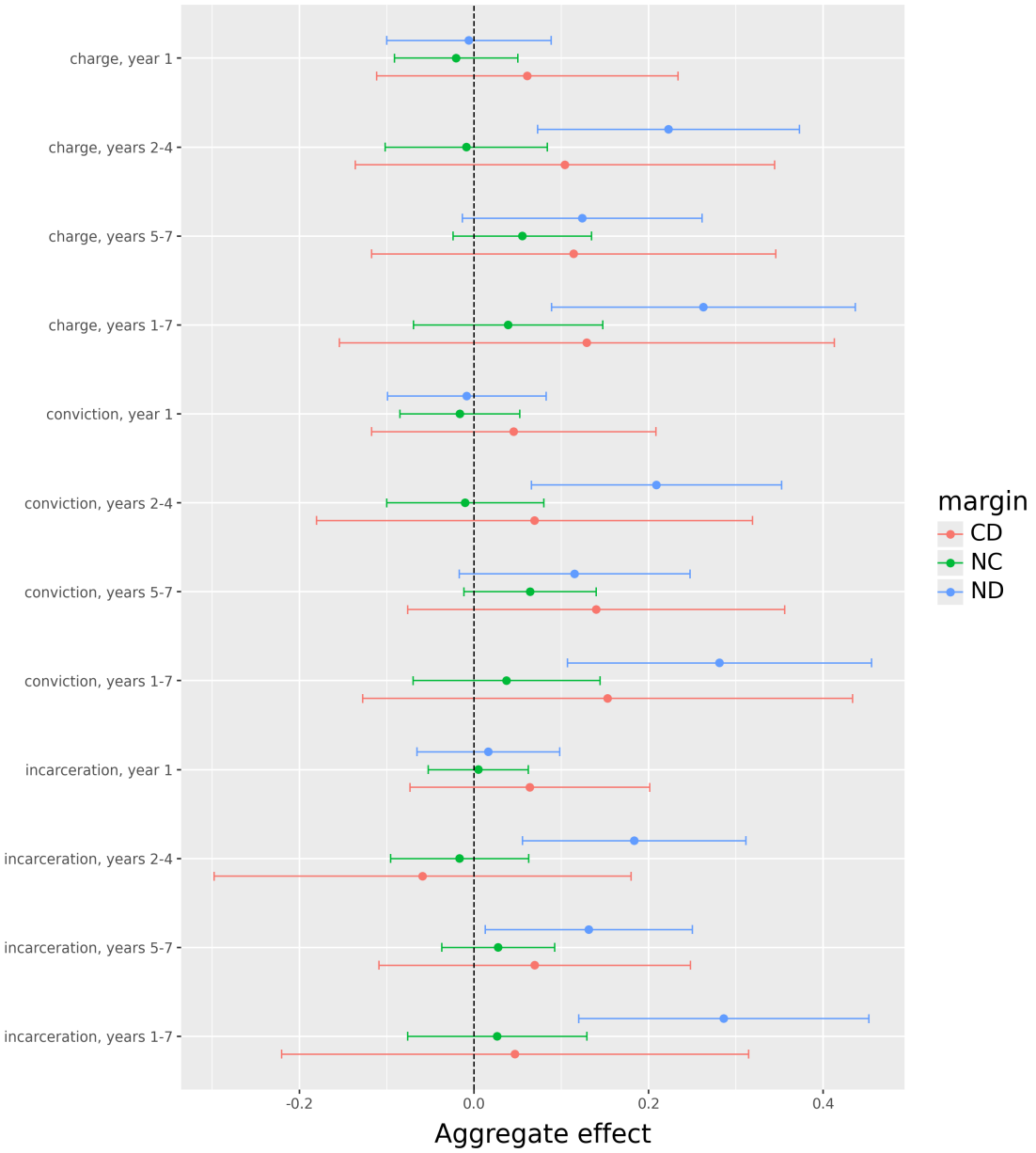
Notes. Each panel reports the cell average $\bar{\tau}_{ts}(\mathcal{X})$ for the indicated recidivism measure and window, with one row per covariate cell. The margins are labeled as in Figure 2 of the main text. Each bar represents the 95% interval described in the text. Cells fix one binary characteristic at the indicated level and overlap across characteristics.

Table 6: Aggregate Effects by Margin and Horizon: Semiparametric Second Stage with Standard Errors

	Year 1	Years 2–4	Years 5–7	Years 1–7
Panel A: CD margin				
New felony charge:	0.066	0.064	0.104	0.101
	(0.080)	(0.102)	(0.104)	(0.131)
New felony conviction:	0.053	0.036	0.118	0.115
	(0.076)	(0.105)	(0.099)	(0.129)
New incarceration:	0.073	−0.065	0.056	0.030
	(0.064)	(0.098)	(0.082)	(0.120)
Panel B: NC margin				
New felony charge:	−0.025	−0.008	0.065*	0.044
	(0.031)	(0.039)	(0.036)	(0.048)
New felony conviction:	−0.021	−0.005	0.068*	0.040
	(0.029)	(0.038)	(0.035)	(0.047)
New incarceration:	−0.004	−0.008	0.039	0.036
	(0.024)	(0.034)	(0.030)	(0.045)
Panel C: ND margin				
New felony charge:	−0.027	0.193***	0.145**	0.238***
	(0.044)	(0.067)	(0.061)	(0.079)
New felony conviction:	−0.018	0.193***	0.129**	0.270***
	(0.042)	(0.064)	(0.059)	(0.079)
New incarceration:	0.002	0.167***	0.143***	0.262***
	(0.038)	(0.057)	(0.054)	(0.075)

Notes. Each entry is the aggregate effect $\bar{\tau}_{ts}$ over the interior region for the indicated recidivism measure and horizon, with the standard error of Section G.4 in parentheses beneath. The last column cumulates over years one to seven. The estimates and the intervals of Figure 2 are those of this table. $^{*}\,p < 0.10$, $^{**}\,p < 0.05$, $^{***}\,p < 0.01$.

Figure 6: Aggregate Effects by Margin and Horizon: Richer Second-Stage Basis



Notes. As in Figure 2, with the coefficient functions approximated by cubic tensor B-splines with five basis functions per axis, one interior knot at the midpoint of the support of each fitted index, in place of the four of the baseline.

Table 7: Judge Randomization: Correlation of Judge-Specific Coefficients with Case Characteristics

	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Panel A: Conviction axis									
(0)	0.005	0.005	-0.014	0.005	0.010	-0.000	-0.001	-0.011	0.001
(1)	-0.003	0.008	0.005	0.005	-0.011	-0.000	0.001	-0.002	-0.002
(2)	0.005	-0.003	-0.001	-0.007	0.011	0.017	-0.002	0.006	0.016
(3)	0.000	-0.001	0.010	-0.000	-0.007	-0.008	0.011	-0.007	-0.003
(4)	-0.003	0.008	-0.007	0.006	-0.007	-0.025	-0.007	-0.012	-0.023
(5)	-0.004	0.000	-0.001	0.000	-0.003	-0.009	-0.002	-0.007	-0.006
(6)	-0.001	-0.002	-0.001	-0.003	0.005	-0.005	0.001	-0.002	-0.006
(7)	0.000	0.006	-0.005	0.012	-0.009	0.001	0.006	0.005	0.002
(8)	-0.010	0.010	-0.004	0.010	-0.023	-0.034	-0.009	-0.017	-0.037
(9)	-0.001	0.002	0.002	0.003	-0.007	-0.004	0.000	0.001	-0.005
Panel B: Incarceration axis									
(0)	0.009	-0.005	-0.002	-0.003	0.015	0.021	0.011	0.021	0.020
(1)	-0.005	0.004	-0.002	0.001	-0.006	-0.011	-0.006	-0.010	-0.015
(2)	0.008	-0.003	-0.007	-0.016	0.027	0.030	-0.011	0.002	0.028
(3)	0.005	-0.002	0.001	0.001	0.005	0.003	0.004	-0.004	0.005
(4)	0.002	0.004	-0.002	0.007	-0.000	0.003	-0.003	0.002	0.003
(5)	0.005	-0.003	0.004	0.007	-0.004	0.019	0.006	0.006	0.021
(6)	0.003	-0.000	-0.004	0.004	0.001	0.002	0.003	-0.001	0.000
(7)	-0.000	0.002	-0.003	0.003	-0.002	0.000	0.003	0.002	-0.002
(8)	-0.005	0.006	-0.001	0.010	-0.021	-0.026	-0.005	-0.004	-0.026
(9)	-0.002	-0.002	0.006	-0.001	-0.005	-0.004	0.004	0.008	-0.003

Notes: Each entry is the sample correlation, across the 256,475 cases, between an estimated judge coefficient in $\hat{\gamma}^t_{J_i}$ (row) and a case characteristic (column) per axis t. Circuit indicators are partialled out of both the coefficients and the characteristics, so the correlations are within circuit. Rows and columns share the same indexing: (1) Female, (2) Black, (3) Any violent charge, (4) Any property charge, (5) Any drug charge, (6) Prior conviction, (7) Number of charges, (8) Misdemeanor charges, (9) Time since prior conviction. Row (0) is the intercept, which has no column counterpart.

the unobserved components is restricted.

As an alternative to their main 2SLS specification, HOSSvD estimate a restricted, judge-fixed-effects version of the same model, fitted separately for each circuit and year: for judge j in a given cell, the latent indices are

$$X_i'\gamma^t_{\text{circ}\times\text{year}} + \alpha^t_{j\times\text{circ}\times\text{year}} - U^t_i, \qquad t \in \{1,2\}, \tag{H.1}$$

where (U^1_i, U^2_i) is specified as a logistic error. Each cell is indexed by a circuit and a year, but is estimated on the three years centered at that year, so the windows of consecutive years overlap in two of their three years. The intercept in $\gamma^t_{\text{circ}\times\text{year}}$ and the level of $\alpha^t_{j\times\text{circ}\times\text{year}}$ are indistinguishable within a cell, so only their sum is identified. They use this sum as the instrument for cases handled by judge j in that circ $\times$ year cell. The specification carries no year, month, or day-of-week term, so these effects are pooled within each circuit-by-year cell. HOSSvD use a mixed logit as the main specification. We impose the plain logit here, which is computationally simpler, and their Appendix shows that the second-stage estimates change little between the two.[15] HOSSvD construct the thresholds within each judge-by-circuit-by-window cell as the log of the 1 and 2 shares of the judge's cases relative to the 0 share.[16] We instead estimate (H.1) by maximum likelihood. The slope structure of (H.1) and that of our augmented specification are not strictly nested: we decompose the slope on X_i additively into a time-invariant judge component γ^t_j and a common year component γ^t_{year} shared across judges, whereas HOSSvD let it vary by circuit and window but not by judge. It is possible to allow for circuit-by-window-level shifts in our model, but it appears unreachable given the current sample size and the number of parameters it would introduce.

HOSSvD's judge enters the instrument only as a level shift, uniform across his or her cases, so he or she contributes a discrete, one-dimensional source of propensity variation. Re-estimating the same judge separately in each circuit-by-year cell is therefore called for in margin-specific effect estimation: the judge is refitted cell by cell, so a judge carries a distinct value in every year he or she serves, which enriches otherwise limited variation. This comes at the cost of the sample available per cell relative to its pooled alternatives, exacerbating attenuation bias concerned with the first-stage estimates. Our semiparametric

[15] See HOSSvD's Online Appendix F and Tables F.1 and F.2, which report the alternative approach under a plain logit and under a diagonal random-effects specification and find results broadly similar to the correlated mixed logit.

[16] See their Online Appendix F.2 and their replication code.

specification lifts both restrictions of (H.1): the cell-common slope $\gamma^t_{\text{circ}\times\text{year}}$ now varies with judge and the logit error gives way to an unrestricted distribution of U^t_i. To separate the two, we introduce an intermediate specification, the *logit*, which adopts the judge-specific slopes but holds the propensity function at the multinomial logit form (5). Reading from HOSSvD to the logit isolates the judge-specific slopes, and with them the treatment-specific variation on which the monotonicity of Section 2 rests, whereas reading from the logit to the semiparametric model isolates the distributional restriction.

The logit fixes the propensity at (5) and is estimated by maximum likelihood. The judge-fixed-effects model is also estimated via maximum likelihood cell by cell with rolling three-year windows.

HOSSvD's specification does not deliver a single likelihood comparable to the others, since it is fitted cell by cell on separate subsamples. The logit and the semiparametric model are both estimated on the full sample and admit a single log-likelihood, compared in Section 6.2.

Figure 7 descends from the cross-judge densities to individual judges, comparing the two fits coefficient by coefficient with their 95% intervals for the judges at the 20th and 80th percentiles of caseload. Notably, the semiparametric intervals are not substantially wider than the logit's despite imposing no distributional assumption. The Monte Carlo experiments of Section 7, calibrated to this application, support taking the comparison at face value: for both fits the estimated standard errors correctly track the sampling dispersion of the coefficients across replications, and when the errors are logistic the semiparametric estimator's sampling variance comes close to the parametric MLE's. When the errors are instead normal, the semiparametric estimator remains accurate while the logit is biased.

H.2 Judge Homogeneity of the First-Stage Coefficients

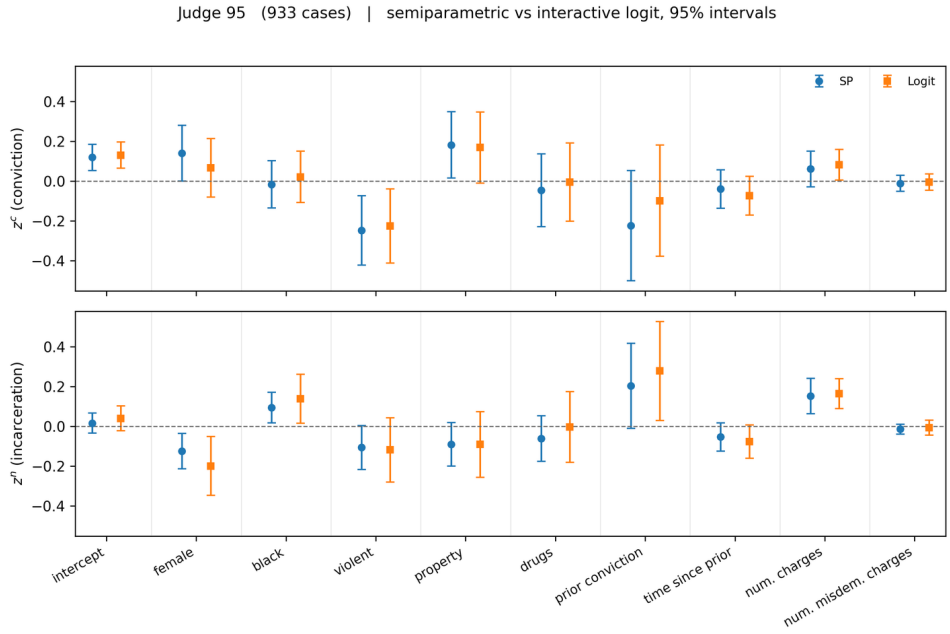
In (H.1) the judge enters the index only through a level shift, and the specification of Section 6.2 lets the slope on X_i vary across judges as well. This section tests the restriction coefficient by coefficient.

Write $\gamma^t_{j,\iota}$ for the ι-th component of the judge block γ^t_j, with $\iota = 1$ the intercept. The null for a component and an axis is

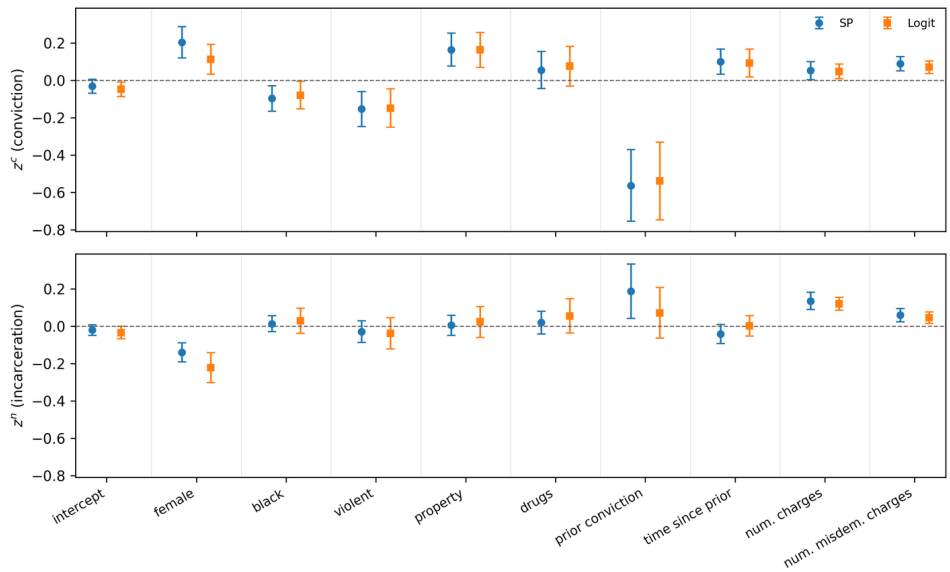
$$H_0(\iota, t) : \gamma^t_{1,\iota} = \gamma^t_{2,\iota} = \cdots = \gamma^t_{J,\iota}, \qquad \iota = 1, \ldots, \dim(X), \quad t \in \{1, 2\}.$$

Figure 7: Judge-Level Coefficient Estimates: Semiparametric vs. Logit

(a) Judge at the 20th caseload percentile



(b) Judge at the 80th caseload percentile



Notes. Each panel pair reports one judge's estimated coefficients on the conviction and incarceration indices under the semiparametric and logit fits, with 95% intervals. The two judges are those at the 20th and 80th percentiles of caseload. Both fits are placed on the common gauge of Figure 1 before plotting.

The statistic is the Wald statistic of the $J-1=128$ deviations $\gamma^t_{j,\iota}-J^{-1}\sum_{j'=1}^{J}\gamma^t_{j',\iota}$ from the cross-judge mean, formed on the block of $\widehat{\mathrm{Var}}(\hat{\theta})$ of Section G.3 that carries those coordinates and referred to the χ^2 distribution with 128 degrees of freedom. The estimator centers the judge intercepts across judges on each axis, and $H_0(1,t)$ is accordingly the statement that every judge intercept is zero, which is the absence of cross-judge variation in the level of the index.

Table 8 reports the twenty statistics with the Holm and Benjamini–Hochberg adjustments across them, together with the joint test over the nine case characteristics on each axis. The two intercept rows carry 11.0 and 7.9 per degree of freedom, and the eighteen hypotheses on the case characteristics run from 1.2 to 3.1. At the five percent level eighteen of the twenty reject under the Benjamini–Hochberg adjustment and fifteen under Holm. The hypotheses not rejected under Benjamini–Hochberg are the female loading and the black loading on the incarceration index. Under Holm the female loading on the conviction index and the two loadings on time since prior conviction are not rejected either. Jointly over the nine characteristics the statistic is 2,438.8 against 1,152 restrictions on the conviction index and 1,365.3 on the incarceration index, and 7,217.0 against 2,304 restrictions over the two axes together.

The cross-judge dispersion of the estimates exceeds the average estimation error on every row. On the conviction index the standard deviation of the estimated loading on prior conviction across the judges is 0.196 against a root mean squared standard error of 0.126, a difference of variances corresponding to a standard deviation of 0.151. The same three numbers for the female loading on that axis are 0.065, 0.059 and 0.027.

The χ^2 reference is an approximation at these dimensions, and the ratio of each joint statistic to its degrees of freedom is 2.12 on the conviction index, 1.19 on the incarceration index and 3.13 over the two axes together.

H.3 Additional Monte Carlo Results

Figures 12 and 13 report further cell averages in the format of Figure 10. Under DGP 1, with the logit correctly specified, both estimators center near the oracle in the two cells of that figure. The deviations that remain are at most 0.024 in absolute value for either estimator against true effects up to 0.31. They stay below 0.4 across-replication standard deviations throughout. Under DGP 2 the two remaining offense-type cells extend the pattern of Section 7. For cases with a violent offense the logit bias sits on the NC margin,

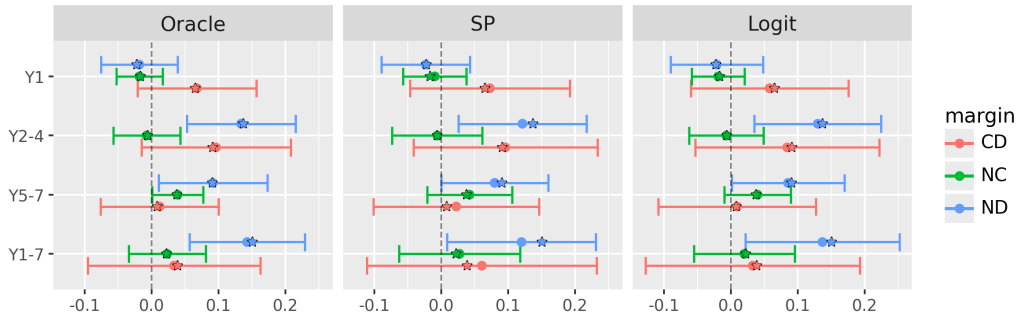
Table 8: Judge Homogeneity of the First-Stage Coefficients

	Conviction index z^1			Incarceration index z^2		
	W	Holm p	BH p	W	Holm p	BH p
Intercept	1,406.8	< 0.001	< 0.001	1,011.0	< 0.001	< 0.001
Female	158.2	0.144	0.042	150.0	0.144	0.090
Black	187.7	0.004	< 0.001	155.2	0.144	0.054
Any violent charge	173.8	0.031	0.006	172.7	0.031	0.007
Any property charge	213.9	< 0.001	< 0.001	218.1	< 0.001	< 0.001
Any drug charge	254.0	< 0.001	< 0.001	327.7	< 0.001	< 0.001
Prior conviction	273.6	< 0.001	< 0.001	279.5	< 0.001	< 0.001
Number of charges	315.8	< 0.001	< 0.001	401.9	< 0.001	< 0.001
Misdemeanor charges	316.5	< 0.001	< 0.001	253.2	< 0.001	< 0.001
Time since prior conviction	166.3	0.064	0.016	156.4	0.144	0.049
All nine characteristics jointly	2,438.8	< 0.001		1,365.3	< 0.001	

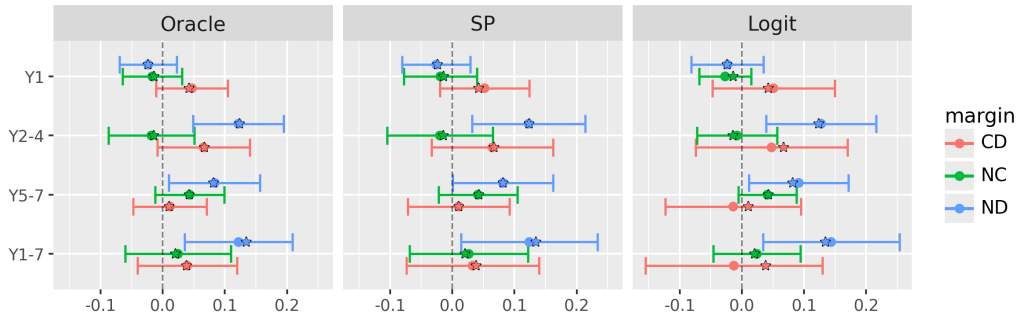
Notes. Each entry in the first ten rows is the Wald statistic of the $J - 1 = 128$ deviations of the judge-specific coefficient in the row from its cross-judge mean, on the index in the column heading, with 128 degrees of freedom, together with the Holm and Benjamini–Hochberg adjustments of its χ^2 p-value across the twenty hypotheses of the table. The last row is the joint test of the nine case characteristics on that index, 1,152 restrictions, with its unadjusted χ^2 p-value. The sample is the 256,475 cases and 129 judges of Section 6.2.

Figure 8: Aggregate Effects across Replications: Semiparametric, Logit, and the Oracle

(a) DGP 1, new felony charge



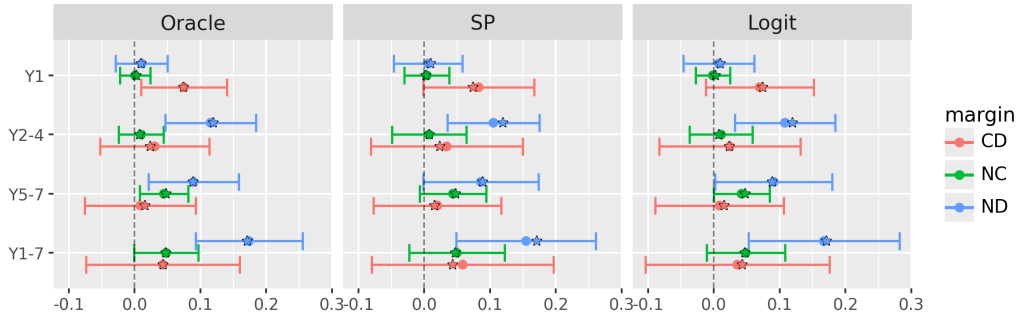
(b) DGP 2, new felony charge



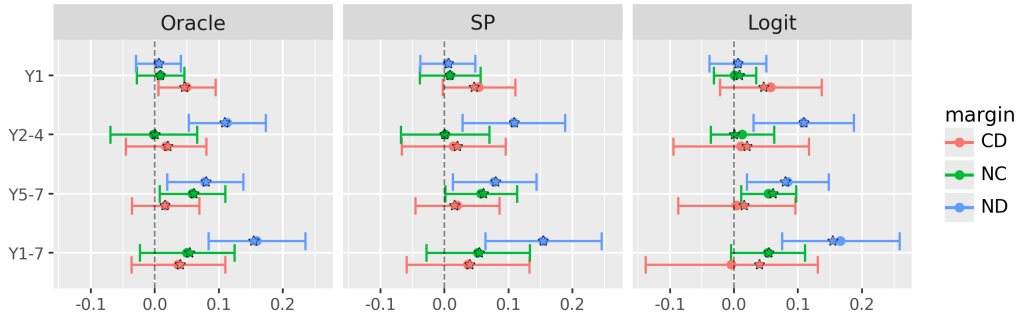
Notes. Each panel displays, for the indicated design and recidivism measure, sampling distributions of the estimated aggregate effects across the 100 replications, by margin and horizon, for the oracle benchmark and the corrected semiparametric and logit estimators. The oracle benchmark observes the true index and requires no correction. Each band marks the mean at the center point and the ± 1.96 standard deviations at the end points. The margins are labeled CD for noncarceral conviction relative to dismissal, NC for incarceration relative to noncarceral conviction, and ND for incarceration relative to dismissal. The star marks the aggregate formed with the true effect function and the weight of the true index.

Figure 9: Aggregate Effects on New Incarceration across Replications: Semiparametric, Logit, and the Oracle

(a) DGP 1, new incarceration



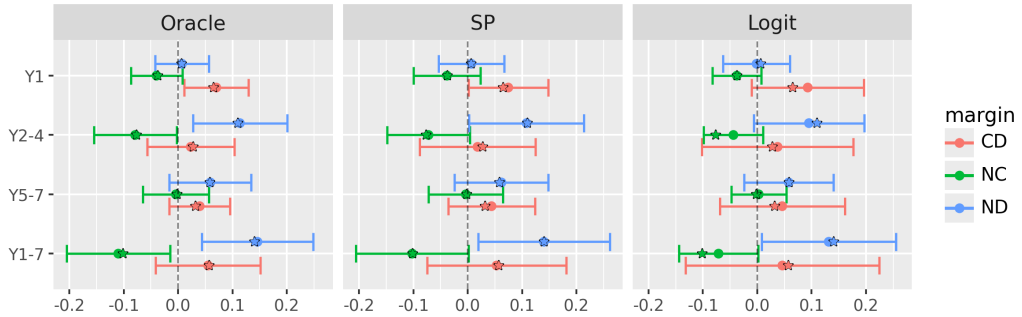
(b) DGP 2, new incarceration



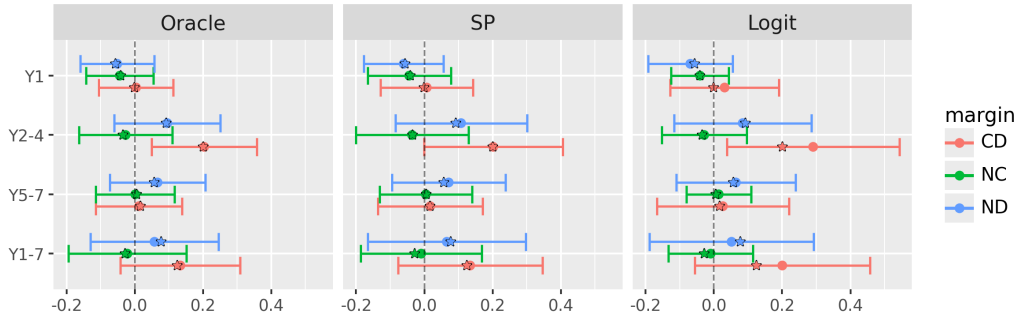
Notes. As in Figure 8, for new incarcerations.

Figure 10: Cell-Average Effects under DGP 2: Semiparametric, Logit, and the Oracle

(a) Cases with a drug offense, new incarceration



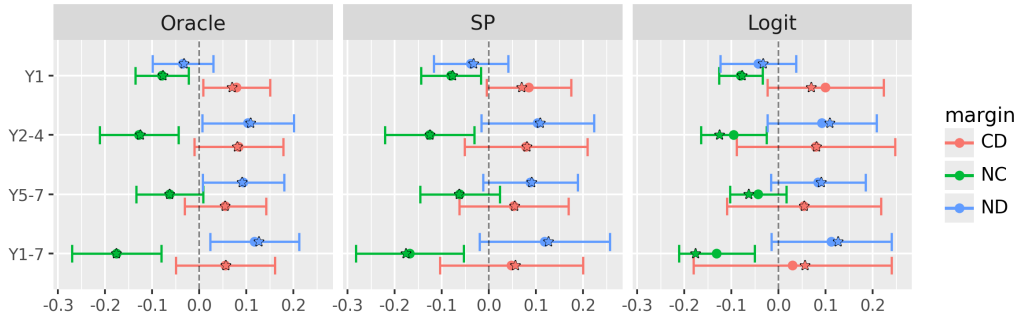
(b) Cases with a prior conviction, new felony charge



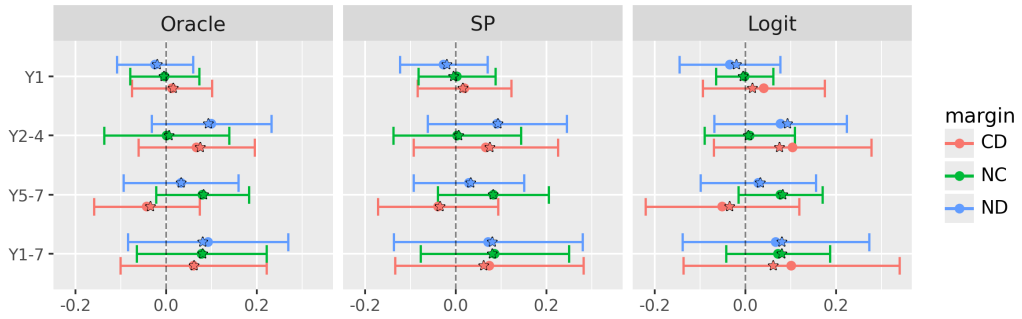
Notes. As in Figure 8, for the average effects over the cell of cases with a drug offense and cases with a prior conviction across the 100 replications of DGP 2. The star marks the truth.

Figure 11: Cell-Average Effects under DGP 2, Remaining Panels: Semiparametric, Logit, and the Oracle

(a) Cases with a drug offense, new felony charge



(b) Cases with a prior conviction, new incarceration



Notes. As in Figure 10, for new felony charges of cases with a drug offense and new incarcerations of cases with a prior conviction.

Table 9: Coverage Rate and Confidence Interval Length Across Replications: Semiparametric Second Stage

	DGP 1			DGP 2		
Outcome	$(1,0)$	$(2,1)$	$(2,0)$	$(1,0)$	$(2,1)$	$(2,0)$
Chrg., Y1	0.950	0.950	0.920	0.970	0.900	0.950
	(0.222)	(0.091)	(0.123)	(0.149)	(0.107)	(0.113)
Chrg., Y2-4	0.990	0.940	0.910	0.930	0.940	0.960
	(0.294)	(0.132)	(0.188)	(0.196)	(0.156)	(0.173)
Chrg., Y5-7	0.960	0.940	0.970	0.950	0.980	0.960
	(0.262)	(0.121)	(0.176)	(0.173)	(0.143)	(0.161)
Chrg., Y1-7	0.970	0.880	0.910	0.970	0.940	0.930
	(0.356)	(0.157)	(0.221)	(0.234)	(0.185)	(0.204)
Conv., Y1	0.960	0.980	0.940	0.980	0.930	0.970
	(0.210)	(0.087)	(0.116)	(0.139)	(0.101)	(0.107)
Conv., Y2-4	0.950	0.930	0.910	0.930	0.950	0.960
	(0.283)	(0.128)	(0.183)	(0.187)	(0.150)	(0.168)
Conv., Y5-7	0.980	0.930	0.960	0.970	0.930	0.970
	(0.250)	(0.116)	(0.169)	(0.166)	(0.138)	(0.155)
Conv., Y1-7	0.930	0.950	0.930	0.970	0.980	0.940
	(0.348)	(0.154)	(0.218)	(0.229)	(0.181)	(0.201)
Inc., Y1	0.950	0.940	0.950	0.930	0.930	0.970
	(0.179)	(0.075)	(0.102)	(0.118)	(0.088)	(0.094)
Inc., Y2-4	0.960	0.920	0.990	0.960	0.950	0.950
	(0.246)	(0.113)	(0.164)	(0.162)	(0.135)	(0.150)
Inc., Y5-7	0.970	0.960	0.890	0.970	0.960	0.980
	(0.215)	(0.101)	(0.150)	(0.140)	(0.120)	(0.137)
Inc., Y1-7	0.960	0.930	0.940	0.950	0.950	0.950
	(0.314)	(0.142)	(0.204)	(0.207)	(0.168)	(0.187)

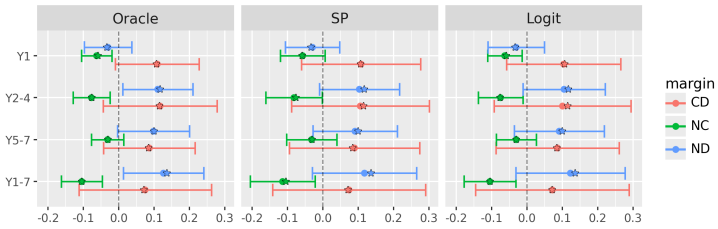
Notes. The outcomes are new charges, convictions, and incarcerations, and the horizons are the first year, years two to four, years five to seven, and the cumulative seven-year window. Each column corresponds to a margin and reports the coverage rate of the 95% confidence intervals from the semiparametric estimator across the 100 replications of the corresponding design, with the average length of those intervals in parentheses beneath. The intervals are those of the corrected estimator, formed by clustered standard errors at the judge-by-year level, when the outcome errors are actually independent across cases.

near 0.04 on new incarcerations over years two to four against a true effect of -0.16, about 0.6 across-replication standard deviations. For cases with a property offense it appears on the CD margin, between -0.04 and -0.06 on both measures over years five to seven and one to seven. The semiparametric biases in these two cells are at most 0.041.

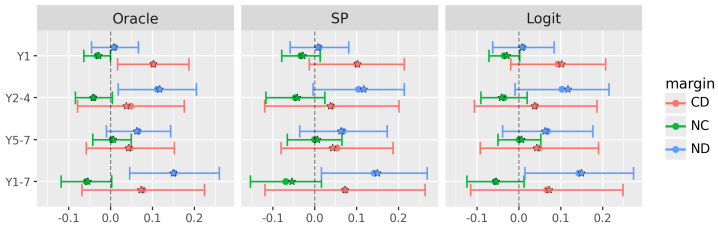
Figure 14 repeats the DGP 1 comparison of Figure 8 with the bias correction switched off. On the cumulative ND rows the uncorrected estimates fall short of the truth by 27 to 36 percent for the semiparametric fit and by 21 to 27 percent for the logit.

Figure 12: Cell-Average Effects under DGP 1: Semiparametric, Logit, and the Oracle

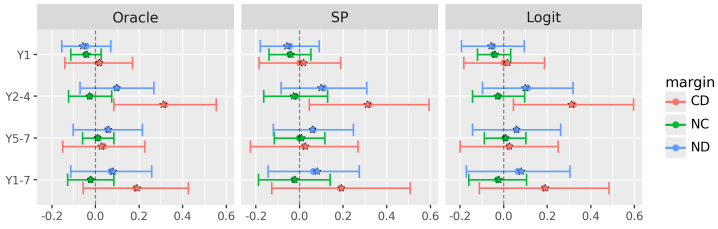
(a) Cases with a drug offense, New felony charge



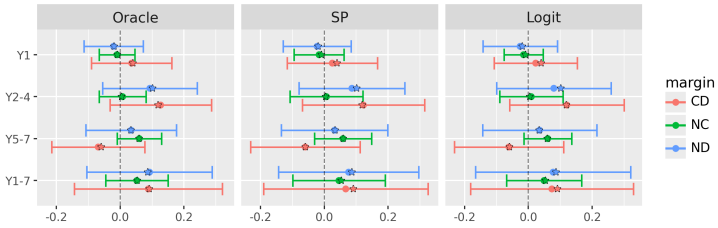
(b) Cases with a drug offense, New incarceration



(c) Cases with a prior conviction, New felony charge



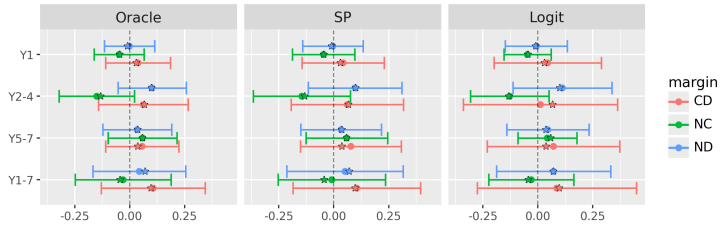
(d) Cases with a prior conviction, New incarceration



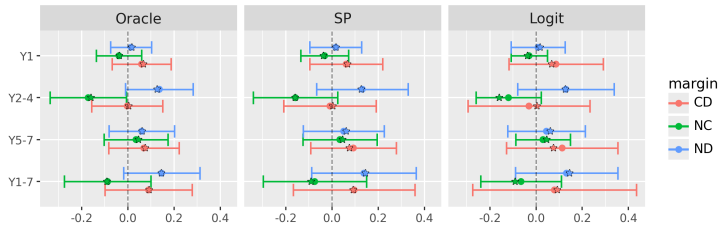
Notes. As in Figure 10, under DGP 1.

Figure 13: Cell-Average Effects under DGP 2 by Offense Type: Semiparametric, Logit, and the Oracle

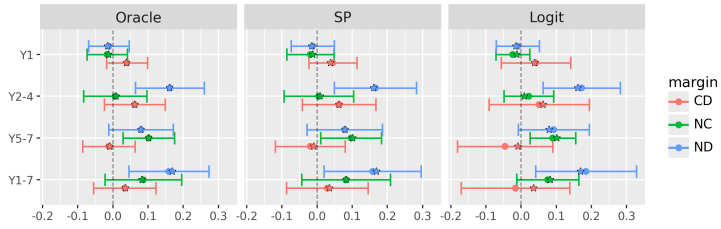
(a) Cases with a violent offense, New felony charge



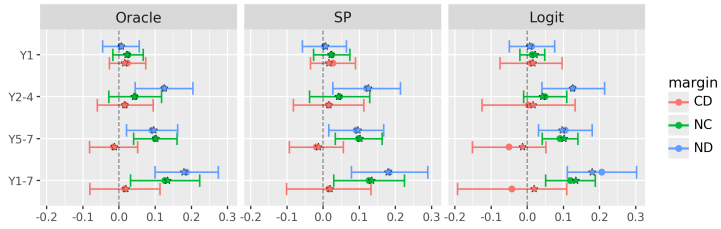
(b) Cases with a violent offense, New incarceration



(c) Cases with a property offense, New felony charge



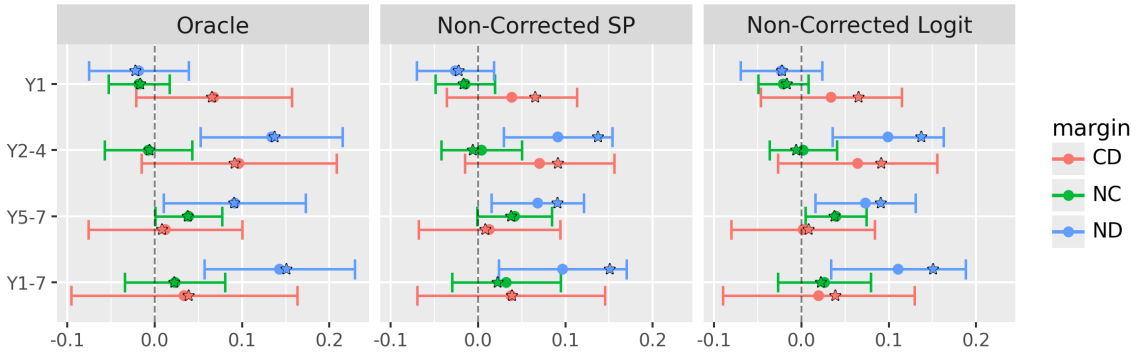
(d) Cases with a property offense, New incarceration



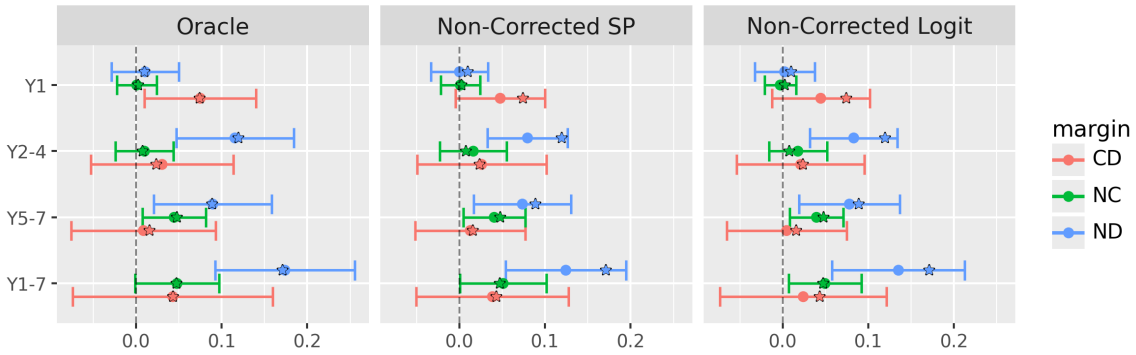
Notes. As in Figure 10, for the cells of cases with a violent offense and with a property offense, under DGP 2.

Figure 14: Aggregate Effects across Replications under DGP 1 without the Attenuation Bias Correction

(a) New felony charge



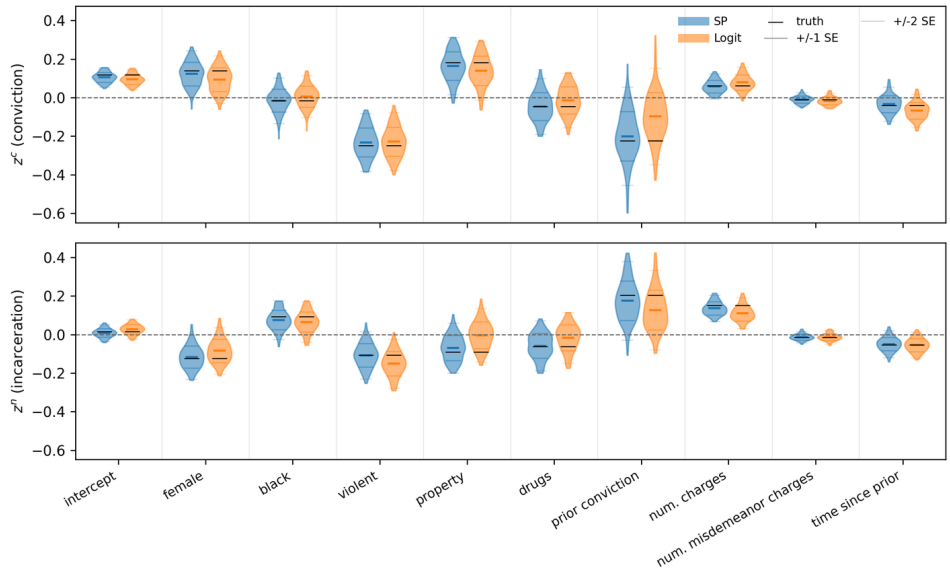
(b) New incarceration



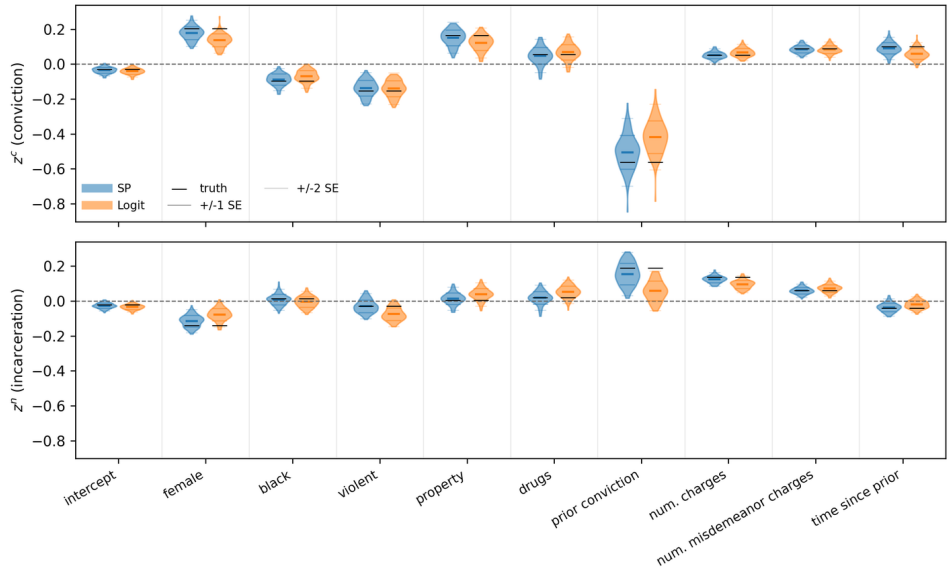
Notes. As in Figure 8, with the second stage estimated without the attenuation bias correction.

Figure 15: Sampling Distributions of Judge-Specific Coefficients under Misspecification of DGP 2

(a) Judge at the 20th caseload percentile



(b) Judge at the 80th caseload percentile



Notes. As in Figure 4.

I Technical Lemmas

Lemma I.1. *The following inequalities hold for all $\theta, \tilde{\theta} \in \Theta$:*

$$\mathsf{L}^2(\theta, \tilde{\theta}) \leq 4\mathsf{H}^2(\theta, \tilde{\theta}) \leq 4\,\mathrm{KL}(\theta \| \tilde{\theta}).$$

Proof of Lemma I.1. First inequality. Let $\theta = (\boldsymbol{\gamma}, f)$ and $\tilde{\theta} = (\tilde{\boldsymbol{\gamma}}, \tilde{f})$. Since $P^t(\cdot \,|\, f)$ is uniformly bounded by 1, it holds that

$$\begin{aligned}
&\sum_{t\in\{0,1,2\}} (P^t(\boldsymbol{\gamma}' W \,|\, f) - P^t(\tilde{\boldsymbol{\gamma}}' W \,|\, \tilde{f}))^2 \\
=&\sum_{t\in\{0,1,2\}} (\sqrt{P^t(\boldsymbol{\gamma}' W \,|\, f)} - \sqrt{P^t(\tilde{\boldsymbol{\gamma}}' W \,|\, \tilde{f})})^2 (\sqrt{P^t(\boldsymbol{\gamma}' W \,|\, f)} + \sqrt{P^t(\tilde{\boldsymbol{\gamma}}' W \,|\, \tilde{f})})^2 \\
\leq& 4 \sum_{t\in\{0,1,2\}} (\sqrt{P^t(\boldsymbol{\gamma}' W \,|\, f)} - \sqrt{P^t(\tilde{\boldsymbol{\gamma}}' W \,|\, \tilde{f})})^2.
\end{aligned}$$

Taking expectation on both sides with respect to the distribution of W yields that $\mathsf{L}(\theta, \tilde{\theta})^2 \leq 4\mathsf{H}(\theta, \tilde{\theta})^2$.

Second inequality. This direction follows from the standard inequality $\mathsf{H}^2(p, q) \leq \mathrm{KL}(p \| q)$ for any two distributions p and q. □

For our purposes, it is convenient to think of the mapping $f \mapsto P^t(z \,|\, f)$, $t \in \{0, 1, 2\}$ as defined on L^1, which gives rise to the signed extension of $P^t(z \,|\, f)$ for f that are not necessarily nonnegative or integrating to one:

$$\begin{aligned}
P^0(z \,|\, f) &= \int_{z^1}^{\infty} \int_{z^2}^{\infty} f(u^1, u^2) du^2 du^1, \\
P^1(z \,|\, f) &= \int_{-\infty}^{z^1} \int_{z^2 - z^1 + u^1}^{\infty} f(u^1, u^2) du^2 du^1, \\
P^2(z \,|\, f) &= \int_{-\infty}^{z^2} \int_{z^1 - z^2 + u^2}^{\infty} f(u^1, u^2) du^1 du^2.
\end{aligned} \tag{I.1}$$

The following lemma provides analytic expressions for the first and second derivatives of $P^t(z \,|\, f)$ with respect to z.

Lemma I.2. *Let f be a continuous integrable function on $\mathbb{R}^2$ satisfying $|f(u)| \leq C(1 + \|u\|)^{-1-\epsilon}$ for all $u \in \mathbb{R}^2$ and some constants $C > 0$ and $\epsilon > 0$. Then the extended treatment propensities are continuously differentiable and admit the following partial derivative*

formulas for $t \in \{0,1,2\}$:

$$
\begin{aligned}
D_1 P^0(z \,|\, f) &= -\int_{z^2}^{\infty} f(z^1, u^2) du^2, \\
D_2 P^0(z \,|\, f) &= -\int_{z^1}^{\infty} f(u^1, z^2) du^1, \\
D_2 P^1(z \,|\, f) &= -\int_{-\infty}^{z^1} f(u^1, z^2 - z^1 + u^1) du^1, \\
D_1 P^2(z \,|\, f) &= -\int_{-\infty}^{z^2} f(z^1 - z^2 + u^2, u^2) du^2.
\end{aligned}
$$

The remaining derivatives can be obtained by the identity $P^0(z \,|\, f) + P^1(z \,|\, f) + P^2(z \,|\, f) = \int_{\mathbb{R}^2} f(u) du$. Moreover, if f is continuously differentiable and $\|\nabla f(u)\| \leq C(1 + \|u\|)^{-1-\epsilon}$, then the extended treatment propensities are twice continuously differentiable and admit the following second-order partial derivative formulas for $t \in \{0,1,2\}$:

$$
\begin{aligned}
D_{11} P^0(z \,|\, f) &= -\int_{z^2}^{\infty} D_1 f(z^1, u^2) du^2, \\
D_{12} P^0(z \,|\, f) &= f(z^1, z^2), \\
D_{22} P^0(z \,|\, f) &= -\int_{z^1}^{\infty} D_2 f(u^1, z^2) du^1, \\
D_{12} P^1(z \,|\, f) &= -f(z^1, z^2) + \int_{-\infty}^{z^1} D_2 f(u^1, z^2 - z^1 + u^1) du^1, \\
D_{22} P^1(z \,|\, f) &= -\int_{-\infty}^{z^1} D_2 f(u^1, z^2 - z^1 + u^1) du^1, \\
D_{11} P^2(z \,|\, f) &= -\int_{-\infty}^{z^2} D_1 f(z^1 - z^2 + u^2, u^2) du^2, \\
D_{12} P^2(z \,|\, f) &= -f(z^1, z^2) + \int_{-\infty}^{z^2} D_1 f(z^1 - z^2 + u^2, u^2) du^2.
\end{aligned}
$$

Proof of Lemma I.2. Throughout, write $\alpha = 1 + \epsilon$, so the hypotheses read $|f(u)| \leq C(1 + \|u\|)^{-\alpha}$ and $|D_l f(u)| \leq C(1 + \|u\|)^{-\alpha}$ for $l = 1, 2$. Both bounds give absolutely integrable integrands in every formula below, justifying the Leibniz interchange.

First-order derivatives. Start from the iterated representation (I.1) of P^0:

$$
P^0(z \,|\, f) = \int_{z^1}^{\infty} F(u^1, z^2) du^1, \qquad F(u^1, z^2) = \int_{z^2}^{\infty} f(u^1, u^2) du^2.
$$

Only the lower limit depends on z^1, so by Leibniz's rule

$$D_1 P^0(z \,|\, f) = -F(z^1, z^2) = -\int_{z^2}^{\infty} f(z^1, u^2) du^2.$$

For $D_2 P^0$, only the lower limit of the inner integral F depends on z^2. The decay bound on f produces an integrable majorant in u^1 uniformly in a neighborhood of z^2, so a second application of Leibniz gives

$$D_2 P^0(z \,|\, f) = \int_{z^1}^{\infty} \partial_{z^2} F(u^1, z^2) du^1 = -\int_{z^1}^{\infty} f(u^1, z^2) du^1.$$

The formulas for $D_2 P^1$ and $D_1 P^2$ follow from the same argument applied to the iterated representations of P^1 and P^2 in (I.1). Each first-order derivative is continuous in z, since f is continuous and the decay bound supplies an integrable majorant for dominated convergence.

Second-order derivatives. Differentiating the first-order formulas a second time, in each case the integrand depends on the differentiation variable through a limit of integration, through the argument of f, or through both. For $D_{11} P^0$, only the argument depends on z^1:

$$D_{11} P^0(z \,|\, f) = D_1 \left(-\int_{z^2}^{\infty} f(z^1, u^2)\, du^2 \right) = -\int_{z^2}^{\infty} D_1 f(z^1, u^2) du^2.$$

For $D_{12} P^0$, only the lower limit of the integral for $D_2 P^0$ depends on z^1, contributing a boundary term:

$$D_{12} P^0(z \,|\, f) = D_1 \left(-\int_{z^1}^{\infty} f(u^1, z^2)\, du^1 \right) = f(z^1, z^2).$$

The remaining displayed identities (for $D_{22} P^0$, $D_{12} P^1$, $D_{22} P^1$, $D_{11} P^2$, $D_{12} P^2$) follow from the same combination of an argument-derivative term and a boundary term. Twice continuous differentiability of $P^t(\cdot \,|\, f)$ follows from continuity of f and ∇f together with the dominated-convergence theorem applied to each Leibniz step. □

Lemma I.3. *Let $f \in \mathcal{F}$. Then $G(\cdot \,|\, f)$ is C^2 on $\mathbb{R}^2$, with*

$$\nabla_z G(z \,|\, f) = (P^1(z \,|\, f), P^2(z \,|\, f)) = P(z \,|\, f), \qquad \nabla_z^2 G(z \,|\, f) = DP(z \,|\, f),$$

where $DP(z \,|\, f)$ is the Jacobian matrix of the propensity function $z \mapsto P(z \,|\, f)$. Moreover,

$DP(z\,|\,f)$ is symmetric and positive definite for all $z \in \mathbb{R}^2$, with

$$D_1P^1(z\,|\,f) > 0, \qquad D_2P^2(z\,|\,f) > 0, \qquad D_2P^1(z\,|\,f) = D_1P^2(z\,|\,f) < 0,$$

and with $D_1P^0(z\,|\,f) < 0$ and $D_2P^0(z\,|\,f) < 0$.

Proof of Lemma I.3. We split the argument into the gradient identity, the Hessian identity together with its symmetry, and the positive-definiteness claim.

Step 1 (Gradient): For each fixed $u \in \mathbb{R}^2$, the integrand

$$\Phi(z, u) = \max_{t\in\{0,1,2\}} (z^t - u^t)$$

is convex and piecewise linear in z. Off the boundary where the argmax changes, whose Lebesgue measure is zero in u at any fixed z, $\Phi(\cdot, u)$ is differentiable with $\partial_{z^1}\Phi(z, u) = \mathbb{1}\{z^1 - u^1 > z^t - u^t,\, t \neq 1\}$. Because f is absolutely continuous with respect to Lebesgue measure, the boundary set has f-measure zero. The difference quotient $h^{-1}[\Phi(z+he_1, u) - \Phi(z, u)]$ is bounded by 1 in absolute value uniformly in u and converges, as $h \to 0$, to $\partial_{z^1}\Phi(z, u)$ for f-almost every u. Dominated convergence with the constant majorant 1 yields

$$\frac{\partial G(z\,|\,f)}{\partial z^1} = \int_{\mathbb{R}^2} \mathbb{1}\{z^1 - u^1 > z^t - u^t,\, t \neq 1\} f(u)du = P^1(z\,|\,f),$$

and analogously $\partial G(z\,|\,f)/\partial z^2 = P^2(z\,|\,f)$. Hence $\nabla_z G(z\,|\,f) = (P^1(z\,|\,f), P^2(z\,|\,f))$ on $\mathbb{R}^2$.

Step 2 (Hessian and symmetry): By definition of $\mathcal{F}$, every $f \in \mathcal{F}$ satisfies the decay bound of Lemma I.2. Consequently, $(P^1(\cdot\,|\,f),\, P^2(\cdot\,|\,f))$ is C^1 on $\mathbb{R}^2$ with bounded derivatives. Combined with $\nabla_z G(\cdot\,|\,f) = (P^1(\cdot\,|\,f), P^2(\cdot\,|\,f))$, this implies that $G(\cdot\,|\,f)$ is C^2 with $\nabla_z^2 G(z\,|\,f)$ equal to $DP(z\,|\,f)$. Symmetry of $DP(z\,|\,f)$ follows from the equality of mixed partials of the C^2 function G.

Step 3 (Positive definiteness):

Using $P^1 = 1 - P^0 - P^2$ and Lemma I.2, we have

$$\begin{aligned} DP_{11}(z\,|\,f) &= D_1P^1(z\,|\,f) = -D_1P^0(z\,|\,f) - D_1P^2(z\,|\,f) \\ &= \int_{z^2}^{\infty} f(z^1, u^2)du^2 + \int_{-\infty}^{z^2} f(z^1 - z^2 + u^2, u^2)du^2. \end{aligned}$$

The change of variable $u^2 = z^2 - z^1 + u^1$ recasts the second integral as

$$\int_{-\infty}^{z^1} f(u^1, z^2 - z^1 + u^1)\, du^1 = -D_2 P^1(z \,|\, f) = |DP_{12}(z \,|\, f)|,$$

where the last equality uses that $DP_{12} = D_2 P^1 \leq 0$ by Lemma I.2. Combining,

$$DP_{11}(z \,|\, f) - |DP_{12}(z \,|\, f)| = \int_{z^2}^{\infty} f(z^1, u^2)\, du^2 > 0,$$

the strict positivity following from the positivity of f at every point of $\mathbb{R}^2$. The symmetric argument gives $DP_{22}(z \,|\, f) - |DP_{21}(z \,|\, f)| > 0$. The change of variable above also gives $-DP_{12}(z \,|\, f) = \int_{-\infty}^{z^1} f(u^1, z^2 - z^1 + u^1) du^1 > 0$, again from $f > 0$. The strict inequalities of the statement follow, those for P^0 from $D_1 P^0 = -(DP_{11} - |DP_{12}|)$ and $D_2 P^0 = -(DP_{22} - |DP_{21}|)$. Hence each Gershgorin disc of $DP(z \,|\, f)$, centered at the diagonal DP_{ii} in each ith row with radius $\sum_{j \neq i} |DP_{ij}|$, lies strictly in the open right half-plane of $\mathbb{C}$. By the Gershgorin circle theorem, every eigenvalue of $DP(z \,|\, f)$ is contained in one of these discs and is therefore real (by symmetry of DP) and strictly positive. Consequently $DP(z \,|\, f)$ is positive definite at every $z \in \mathbb{R}^2$. □

Lemma I.4 (Injectivity of treatment propensities)**.** *Let $f \in \mathcal{F}$. Then the induced propensity function $z \mapsto (P^1(z \,|\, f),\, P^2(z \,|\, f))$ is injective on $\mathbb{R}^2$.*

Proof of Lemma I.4. By Lemma I.3, $G(\cdot \,|\, f)$ is C^2 on $\mathbb{R}^2$ with Hessian $\nabla_z^2 G(z \,|\, f) = DP(z \,|\, f)$ positive definite at every z. A C^2 function whose Hessian is positive definite everywhere is strictly convex, and the strict convexity of $G(\cdot \,|\, f)$ in turn implies strict monotonicity of its gradient: for any $z_1, z_2 \in \mathbb{R}^2$ with $z_1 \neq z_2$,

$$(z_1 - z_2)'(\nabla_z G(z_1 \,|\, f) - \nabla_z G(z_2 \,|\, f)) > 0.$$

Since $\nabla_z G(z \,|\, f) = (P^1(z \,|\, f), P^2(z \,|\, f))$ by Lemma I.3, $z \mapsto (P^1(z \,|\, f), P^2(z \,|\, f))$ is injective on $\mathbb{R}^2$. □

Lemma I.5 (P_o is an open map)**.** *The propensity function P_o is an open map on $\mathbb{R}^2$ and a homeomorphism onto its image. Consequently, for any $A_1, A_2 \subseteq \mathbb{R}^2$,*

$$P_o(\operatorname{Int} A_1 \cap \operatorname{Int} A_2) = \operatorname{Int} P_o(A_1) \cap \operatorname{Int} P_o(A_2).$$

Moreover, if $A \subseteq \mathbb{R}^2$ is closed, every interior point of the closure of $P_o(A)$ is an interior

point of $P_o(A)$.

Proof of Lemma I.5. By Lemma I.3, the Jacobian $DP_o(z) = DP(z \,|\, f)$ is positive definite, hence nonsingular, at every $z \in \mathbb{R}^2$. The inverse function theorem then makes P_o a local diffeomorphism at each point, so it maps open sets to open sets and is therefore an open map. Being injective by Lemma I.4, P_o is a continuous open bijection onto its image, hence a homeomorphism onto it, and so preserves interiors: $P_o(\operatorname{Int} A) = \operatorname{Int} P_o(A)$ for every $A \subseteq \mathbb{R}^2$. Injectivity also gives $P_o(\operatorname{Int} A_1 \cap \operatorname{Int} A_2) = P_o(\operatorname{Int} A_1) \cap P_o(\operatorname{Int} A_2)$, and combining the two identities proves the displayed identity. The last claim rests on the limit $\min\{P_o^0(z), P_o^1(z), P_o^2(z)\} \to 0$ as $\|z\| \to \infty$, which follows from $P_o^0(z) \le \mathbb{P}(U_i^t \ge z^t)$ and $P_o^t(z) \le \mathbb{P}(U_i^t \le z^t)$ for $t \in \{1, 2\}$. Now let A be closed, and let $P_o(z_n)$ with $z_n \in A$ converge to a point of the closure of $P_o(A)$. If (z_n) has a bounded subsequence, a further subsequence converges to some $z \in A$, and the limit point is $P_o(z)$ by continuity. Otherwise $\|z_n\| \to \infty$, and by the limit above the limit point lies on the boundary of the closed triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$, which contains the closure of $P_o(A)$. A boundary point of the triangle is interior to no subset of the triangle, and the interior of the closure of $P_o(A)$ therefore lies in $P_o(A)$ and, as an open set, in $\operatorname{Int} P_o(A)$. □

Lemma I.6 (Positive semidefiniteness and injectivity)**.** *Let $P = (P^1, P^2)$ be a continuously differentiable function on a convex subset of $\mathbb{R}^2$ that satisfies conditions (iii) and (iv) in the definition of $\mathcal{P}$ at every point of its domain, where $P^0 = 1 - P^1 - P^2$. Then the Jacobian $DP(z)$ is symmetric and positive semidefinite at every z in the domain, and is positive definite at every z at which it is nonsingular. Moreover, if $DP(z_0) > 0$ at a point z_0 of the domain, then $P(z_0) \ne P(z)$ for every other point z of the domain. In particular P is injective on $\{z : DP(z) > 0\}$. The hypotheses hold for every $P \in \mathcal{P}$ with domain $\bar{\mathcal{Z}}$ and for the function P of every candidate pair with domain $\mathbb{R}^2$.*

Proof of Lemma I.6. Fix z in the domain of P. Condition (iii) in the definition of $\mathcal{P}$ makes $DP(z)$ symmetric, with $DP_{12}(z) = DP_{21}(z)$. Condition (iv) in the definition of $\mathcal{P}$, applied to P^1, P^2 and P^0, gives $DP_{12}(z) \le 0$, $DP_{11}(z) \ge |DP_{12}(z)|$, and $DP_{22}(z) \ge |DP_{21}(z)|$, so each Gershgorin disc of $DP(z)$, centered at the nonnegative diagonal entry $DP_{ii}(z)$ with radius $\sum_{l \ne i} |DP_{il}(z)|$, lies in the closed right half-plane of $\mathbb{C}$. By the Gershgorin circle theorem and the symmetry of $DP(z)$, every eigenvalue is real and nonnegative, so $DP(z)$ is positive semidefinite. A symmetric positive semidefinite matrix with no zero eigenvalue is positive definite, which gives the second claim. For the separation claim, fix z_0 with $DP(z_0) > 0$ and any z in the domain with $v := z - z_0 \ne 0$. Since the domain is convex, it

contains $z_0 + \vartheta v$ for every $\vartheta \in [0,1]$, and the fundamental theorem of calculus gives

$$v'\left(P(z) - P(z_0)\right) = \int_0^1 v' DP(z_0 + \vartheta v)\, v \, d\vartheta.$$

The integrand is nonnegative because $DP \geq 0$ on the domain, is continuous in ϑ, and equals $v'DP(z_0)\, v > 0$ at $\vartheta = 0$, so the integral is strictly positive. Hence $P(z) \neq P(z_0)$, and applying this with z_0 ranging over the positive-definite set gives injectivity of P there. $\square$